BERNOULLI (BETA) and INTEGER PART SEQUENCES

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1. Introduction

A BERNOULLI or for brevity a Beta sequence for the real number α, which we will represent by $\beta(\alpha)$ or simply β, is defined as the infinite sequence :

$$\beta(\alpha) = \beta =< \beta_k >= \beta_1, \beta_2, \beta_3, \ldots,$$

where, $\beta_k = [(k+1)\alpha] - [k\alpha]$, $k = 1, 2, 3, \ldots,$ and $[x]$ is the integer part of x. (1)

That is β is simple the forward difference of the integer part sequence obtained from multiples of a real number.

Sequences of this type were first studied by Johann Bernoulli III (and hence the name Beta), the astronomer grandson of the famous mathematician Johann Bernoulli I. Although we are unable to identify the exact problems that stimulated his concern for these sequences, it probably had something to do with a cogwheel representation of planetary orbits which resulted in him being forced to calculate the integer parts of large multiples of irrationals. Without a computer this is very time consuming. However knowing the corresponding Beta sequence this becomes trivial using the property shown in P1 below.

Our interest in Beta sequences was aroused after reading some unpublished notes of Douglas Hofstadter [5], which gave a very spirited introduction to these sequences together with many fascinating examples; in particular they contain a description of the INT function which is rather cryptically described in his famous book “Gödel, Escher, Bach".

Johann III observed in 1772 [1], but did not prove, that in such sequences, having calculated the first few terms of the above integer parts formula (1), these terms can then be used to generate a larger number of terms and then this new subsequence can be used to generate an even larger subsequence and so on. Each time the increase in the number of terms is itself increasing which allows us to generate the sequence extremely rapidly. We describe a method for the rapid generation of the β - sequence in detail in section 4 of this paper.

The proof arising out of this observation had to await over a century until 1882 when A. Markov [6] established it using continued fractions (see theorem 3 below). For material relating to continued fractions the reader is advised to consult any text such as Roberts [7]. The relationship between characteristics and the three gap theorem is discussed in van Ravenstein , Winley, and Tognetti [8].

Before examining the properties of β - sequences (section 2) and their associated derived sequences (section 3), we give some examples of the β - sequences associated with particular real numbers.

Examples

(a) $\alpha = \pi = 3.1415\ldots$

k	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
[kα]	3	6	9	12	15	18	21	25	28	31	34	37	40	43	47
β_k	3	3	3	3	3	3	4	3	3	3	3	3	3	4	3

(b) $\alpha = \tau = 1.618\ldots = \frac{\sqrt{5}+1}{2}$

k	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
[kα]	1	3	4	6	8	9	11	12	14	16	17	19	21	22	24
β_k	2	1	2	2	1	2	1	2	2	1	2	2	1	2	1

(c) $\alpha = \sqrt{2} = 1.4142\ldots$

k	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
[kα]	1	2	4	5	7	8	9	11	12	14	15	16	18	19	21
β_k	1	2	1	2	1	1	2	1	1	2	1	2	1	2	1

(d) $\alpha = \mathrm{e} = 2.7182\ldots$

k	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
[kα]	2	5	8	10	13	16	19	21	24	27	29	32	35	38	40
β_k	3	3	2	3	3	3	2	3	3	2	3	3	3	2	3

(e) $\alpha = \mathrm{e}^{-1} = 0.3678\ldots$

k	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
[kα]	0	0	1	1	1	2	2	2	3	3	4	4	4	5	5
β_k	0	1	0	0	1	0	0	1	0	1	0	0	1	0	0

(f) $\alpha = 0.618\ldots = \frac{1}{\tau} = \tau - 1$

k	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
[kα]	0	1	1	2	3	3	4	4	5	6	6	7	8	8	9
β_k	1	0	1	1	0	1	0	1	1	0	1	1	0	1	0

(g) $\alpha = 0.3819\ldots = \frac{1}{\tau^2} = 1 - \frac{1}{\tau}$

k	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
[kα]	0	0	1	1	1	2	2	3	3	3	4	4	4	5	5
β_k	0	1	0	0	1	0	1	0	0	1	0	0	1	0	1

(h) $\alpha = \sqrt{2} - 1 = 0.4142\ldots$

k	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
[kα]	0	0	1	1	2	2	2	3	3	4	4	4	5	5	6
β_k	0	1	0	1	0	0	1	0	1	0	0	1	0	1	0

(i) $\alpha = 1 - (\sqrt{2} - 1) = 2 - \sqrt{2} = 0.5857\ldots$

k	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
[kα]	0	1	1	2	2	3	4	4	5	5	6	7	7	8	8
β_k	1	0	1	0	1	1	0	1	0	1	1	0	1	0	1

2. Properties

In order to simplify the proofs associated with the properties presented here, the reader is referred to the **Appendix** (where results are referred to as A1, A2 etc.). This appendix is a list of properties for the bracket functions $[x]$ and $\{x\}$ (the integer part of x and the fractional part of x respectively).

P1. (a)

$$\sum_{i=1}^{k} \beta_i = [(k+1)\alpha] - [\alpha]$$

Thus $[k\alpha]$ is easily calculated from the sum of the terms in the Beta sequence.

(b)

$$\lim_{k\to\infty} \left(\frac{\sum_{i=1}^{k} \beta_i}{k} \right) = \alpha$$

Proof: (a) This follows immediately from (1).

(b) This follows from (a) and A21.

P2. (a) If α is the rational number n/m then $\beta_k(\alpha) = \beta_{k+m}(\alpha)$. That is the sequence is periodic with period m.

(b) If we know the β sequence for $\{\alpha\}$ we simply add $[\alpha]$ to each term to obtain the β sequence for α.

(c) $\beta_k(\alpha) = -\beta_k(-\alpha)$.

Proof: (a) If $\alpha = n/m$ then

$$\begin{aligned}
\beta_k(\alpha) &= [(k+1)\alpha] - [k\alpha] \\
&= [(k+1)n/m] - [kn/m] \\
&= n + [(k+1)n/m] - n - [kn/m] \\
&= [n + (k+1)n/m] - [n + kn/m], \text{ from A7} \\
&= [(k+m+1)n/m] - [(k+m)n/m] \\
&= \beta_{k+m}(\alpha).
\end{aligned}$$

(b) $\beta_k = [(k+1)\alpha] - [k\alpha] = [(k+1)\{\alpha\}] - [k\{\alpha\}] + [\alpha]$ from A7.

(c) This follows from A11.

Note

From P2(a), we see that the β - sequence for a rational number α is formed by simply repeating the first m terms of the sequence where $\alpha = n/m$.

Consequently such sequences are different in character from those associated with irrational values of α and henceforth, we will consider only those β - sequences for which α is irrational. We emphasise that, for reasons that will become obvious below, we can restrict ourselves to $0 < \alpha < 1$.

Thus in what follows we can confine ourselves to

$$\beta_k = [(k+1)\{\alpha\}] - [k\{\alpha\}]$$

We also note that $\{\alpha\} = \alpha$ has the simple continued fraction expansion

$$\alpha = (0; a_1, a_2, a_3, \ldots) = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \ldots}}}$$

.

P3. If 0< α <1 then β_k takes on only one of the values 0 or 1.

Proof:

$$\begin{aligned} [(k+1)\alpha] &= [k\alpha + \alpha] \\ &= [\{k\alpha\} + [k\alpha] + \alpha] \\ &= [k\alpha] + [\{k\alpha\} + \alpha], \text{ from A7.} \end{aligned}$$

Hence $\beta_k = [y_k]$ where $y_k = \{k\alpha\} + \alpha$. Now $0 \leq y_k < 2$ and so $\beta_k = [y_k] = 0$ or 1.

We see that $\beta_k = 0$ means that $\{k\alpha\} + \alpha < 1$ and that $\beta_k = 1$ means that $\{k\alpha\} + \alpha \geq 1$. Consequently we see, from P2 (b), that β_k takes on only one of the values $[\alpha]$ or $[\alpha] + 1$. We note that this is true for any real value of α.

P4. $\{y_k\} = \{\{k\alpha\} + \alpha\} = \{(k+1)\alpha\}$

Proof:

$$\begin{aligned}\{(k+1)\alpha\} &= \{k\alpha+\alpha\}\\ &= \{[k\alpha]+\{k\alpha\}+\alpha\}\\ &= \{\{k\alpha\}+\alpha\},\ \text{from A8.}\end{aligned}$$

P5. (a) If 0< α <1/2 and $\beta_k = 1$ then $\beta_{k+1} = 0$.

(b) If 1/2< α <1 and $\beta_k = 0$ then $\beta_{k+1} = 1$.

Proof:

$$\begin{aligned}\{(k+1)\alpha\} &= y_k-[y_k],\ \text{from P4}\\ &= y_k-\beta_k,\ \text{from P3}\\ &= \{k\alpha\}+\alpha-\beta_k.\end{aligned}$$

Hence

$$\begin{aligned}y_{k+1} &= \{(k+1)\alpha\}+\alpha\\ &= \{k\alpha\}+2\alpha-\beta_k.\end{aligned}$$

(a) If $0 < \alpha < 1/2$, then $0 < 2\alpha < 1$ and with $\beta_k = 1$ we have $0 < y_{k+1} = \{k\alpha\} + 2\alpha - 1$ and $0 < \{k\alpha\} < 1$, which establishes that, $\beta_{k+1} = [y_{k+1}] = 0$.

(b) If $1 > \alpha > 1/2$, then $2 > 2\alpha > 1$ and with $\beta_k = 0$, we have $2 > y_{k+1} = \{k\alpha\} + 2\alpha > 1$, which establishes that, $\beta_{k+1} = [y_{k+1}] = 1$.

From P5 (a) it is seen that if $0 < \alpha < 1/2$ then units can occur only as isolated singles separating groups of zeros. Similarly from P5 (b) we see that if $1 > \alpha > 1/2$ then zeros can occur only as isolated singles separating groups of units. From now on we refer to the integer that occurs singly as the separator. The other term will be referred to as the string term.

Thus the string term is equal to the nearest integer to α, namely $[\alpha + 1/2]$ (see A19). If this is $[\alpha]$ then $0 < \{\alpha\} < 1/2$ and the separator is $[\alpha] + 1$. Otherwise the string term is $[\alpha] + 1$ and $1/2 < \{\alpha\} < 1$ and the separator if $[\alpha]$.

This is consistent with the mean behaviour of the sequence. For example, when $0 < \alpha < 1/2$ we would expect more zeros than units, that is strings of zeros separated by single occurrences of units.

P6. If $\gamma = 1 - \alpha$ and $0 < \alpha < 1$, then $\beta(\gamma)$ is obtained from $\beta(\alpha)$ by replacing the zeros by units and the units by zeros.

Proof:

$$\begin{aligned}
\beta_k(\alpha) &= [(k+1)\alpha] - [k\alpha] \\
&= [(k+1)(1-\gamma)] - [k(1-\gamma)] \\
&= k+1+[-(k+1)\gamma] - k - [-k\gamma], \text{ from A7} \\
&= 1+[-(k+1)\gamma] - [-k\gamma] \\
&= 1+(-1-[(k+1)\gamma]) - [-1-[k\gamma]), \text{ from A11} \\
&= 1-\beta_k(\gamma).
\end{aligned}$$

P7. If $0 < \alpha < 1$ then the number of terms between the kth and $(k+1)$th separators in $\beta(\alpha)$ is $p_{k+1} - p_k - 1$ where

$$p_k = \begin{array}{ll} [k/\alpha], & 0<\alpha<1/2, \\ [k/(1-\alpha)], & 1/2<\alpha<1. \end{array}$$

Proof: If $0 < \alpha < 1/2$ then the separators in $\beta(\alpha)$ are units and there are integers p_k and p_{k+1} such that $p_k\alpha < k < (p_k+1)\alpha$ and $\alpha p_{k+1} < k+1 < (p_{k+1}+1)\alpha$, which means that $p_k = [k/\alpha]$.

Now $[(p_k+1)\alpha] \geq k > [p_k\alpha]$ and so $\beta_{p_k} > 0$. Hence from P3 we have $\beta_{p_k} = 1$ and similarly $\beta_{p_{k+1}} = 1$.

Since

$$\sum_{i=p_k+1}^{p_{k+1}-1} \beta_i(\alpha) = [p_{k+1}\alpha] - [(p_k+1)\alpha]$$

we see from A23 and P3 that

$$\beta_{p_k+1}(\alpha) = \beta_{p_k+2}(\alpha) = \ldots = \beta_{p_{k+1}-1}(\alpha) = 0$$

Thus the number of terms between the kth and $(k+1)$th separators in $\beta(\alpha)$ is

$$p_{k+1} - p_k - 1 = [(k+1)/\alpha] - [k/\alpha] - 1.$$

If $1/2 < \alpha < 1$ then $0 < 1-\alpha = \gamma < 1/2$ and from P6 we know that the number of terms between the kth and $(k+1)$th separators in $\beta(\alpha)$ is the same as in $\beta(\gamma)$. Hence the number of terms between the kth and $(k+1)$th separators in $\beta(\alpha)$ is

$$[(k+1)/\gamma] - [k/\gamma] - 1 = [(k+1)/(1-\alpha)] - [k/(1-\alpha)] - 1.$$

We note that P7, with α replaced by $\{\alpha\}$, is true for any irrational number α.

P8. We note that **Graham, Knuth and Patashnik [4]** refer to the sequence $[k/\alpha], k = 1, 2, \ldots$ as Spec(α) and obtain several of the properties we have described. Additionally they show that the number of terms in Spec(α) that are less than equal to n is $[(n+1)/\alpha]$.

3. Derived Sequences

We now form a new sequence $\beta'(\alpha)$ (the derived sequence) by making each new term equal to the number of terms in a run of the terms between two consecutive separators in the sequence $\beta(\alpha)$.

Consider for example, $\beta(e^{-1}) = 0, 1, 0, 0, 1, 0, 0, 1, 0, 1, 0, 0, 1, \ldots$. We note that e^{-1} is between 0 and 1 with 0 the nearest integer. Hence as is usual for such a β sequence there are runs of 0's between units as separators starting with the first unit. We note that there are two terms in this sequence between the first and second separators and there are two terms between the second and third separators with one term between the next consecutive pair of separators and so on. In this way the derived sequence is found to be

$$\beta'(e^{-1}) = 2, 2, 1, 2, 2, 2, 1, 2, 2, \ldots$$

From inspection it appears that this derived sequence may be another β - sequence with 1 as the separator and 2 as the string term. We now show that this is indeed the case and that the result holds in general.

Theorem 1

For any irrational number α, $\quad \beta'(\alpha) = \beta(\alpha')$

$$\begin{aligned} \text{where } \alpha' \quad &= z, \qquad \{\alpha\} < 1/2, \\ &= 1/z, \qquad \{\alpha\} > 1/2, \\ \text{and } \quad z \quad &= 1/\{\alpha\} - 1. \end{aligned}$$

Proof: If we agree to count the number of terms between consecutive separators in $\beta(\alpha)$ beginning with the first occurrence of a separator in $\beta(\alpha)$, then with p_k as defined in P7, we see that the kth term in the derived sequence $\beta'(\alpha)$ is, from P7,

$$\beta'_k(\alpha) = p_{k+1} - p_k - 1.$$

If $0 < \{\alpha\} < 1/2$ this becomes

$$\begin{aligned} \beta'_k(\alpha) &= [(k+1)/\{\alpha\}] - [k/\{\alpha\}] - (k+1) - k \\ &= [(k+1)(1/\{\alpha\} - 1)] - [k(1/\{\alpha\} - 1), \text{ from A7.} \end{aligned}$$

Thus $\beta'_k(\alpha)$ is the kth term of the β - sequence for $1/(1/\{\alpha\}-1)$. Alternatively if $1/2 < \{\alpha\} < 1$ then,

$$\begin{aligned}\beta'_k(\alpha) &= [(k+1)(1/(1-\{\alpha\})-1)] - [k(1/(1-\{\alpha\})-1)] \\ &= [(k+1)(1/(1/\{\alpha\}-1))] - [k(1/(1/\{\alpha\}-1))].\end{aligned}$$

Thus $\beta'_k(\alpha)$ is the kth term of the β - sequence for $1/(1/\{\alpha\}-1)$.

Corollary 1

The number of terms in the strings between two consecutive separators in a β - sequence must be one of only two values $[\alpha']$ or $[\alpha']+1$.

Proof: This follows immediately from Theorem 1 and the comments following P5.

Corollary 2

a) If $0 < \{\alpha_1\} < \{\alpha_2\} < 1/2$ then $\alpha'_1 > \alpha'_2 > 1$.

a) If $1/2 < \{\alpha_1\} < \{\alpha_2\} < 1$ then $\alpha'_2 > \alpha'_1 > 1$.

Proof: The proof of both parts follows directly from Theorem 1.

If we graph α' against $\{\alpha\}$, then the properties in Corollary 2 are seen clearly from the graph and we note that the graph is symmetrical about the line $\{\alpha\} = 1/2$.

Derived Sequences and Continued Fractions

We now consider the relationship between the terms in the derived sequence and the terms in the simple continued fraction expansion of α.

Suppose $\{\alpha\} < 1/2$ where $\{\alpha\}$ has the **simple continued fraction** expansion $\{0; a_1, a_2, a_3, \ldots\}$. Then from the theory of continued fractions and Theorem 1 we have,

$$\alpha' = z = 1/\{\alpha\} - 1 = \{a_1 - 1; a_2, a_3, \ldots\} > 1 \text{ and } a_1 \geq 2.$$

Hence, when $\{\alpha\} < 1/2$ the terms in the derived sequence $\beta'(\alpha)$ must take on the values $[1/\{\alpha\}-1]$ or $[1/\{\alpha\}]$, that is $a_1 - 1$ or a_1.

For example, if $2/5 < \{\alpha\} < 1/2$ then $1 < \alpha' < 3/2$ and so $\{\alpha'\} < 1/2$. Specifically, $\{\sqrt{2}\}$ lies in the interval (0.4, 0.5) and $\{\sqrt{2}\}$ has the continued fraction expansion $\{0; 2, 2, 2, \ldots\}$. Consequently $\alpha' = 1/\{\sqrt{2}\} - 1 = \sqrt{2}$ has the continued fraction expansion

$\{1; 2, 2, 2, \ldots\}$ and the terms in the derived sequence take on the values 1 and 2. It is seen that the continued fraction expansion for α' amounts to a left shift by one term in the continued fraction expansion for $\sqrt{2}$ and a change in the first occurrence of a 2 to a 1. We also note that in this example the derived sequence is identical to the β - sequence for $\sqrt{2}$ (see section 1, example (c)).

On the other hand $\{\alpha\} > 1/2$ implies that $a_1 = 1$ and since $1 < 1/\{\alpha\} < 2$ we see that $z = 1/\{\alpha\} - 1 = \{0; a_2, a_3, \ldots\} < 1$ consequently $\alpha' = 1/z = \{a_2; a_3, \ldots\} > 1$ and thus the terms in the derived sequence take on the values a_2 or $a_2 + 1$.

From the above for any α we can inspect the terms of its continued fraction expansion $\{a_0; a_1, a_2, \ldots\}$ and determine which terms will be in the derived sequence for α as follows.

We ignore the initial term a_0. If $a_1 \geq 2$ then the derived sequence will take on the values $a_1 - 1$ or a_1. On the other hand, if $a_1 = 1$ the terms in the derived sequence will be either $a_2 + 1$ or a_2.

From this it is seen that a derived sequence can never have zero as a term. Consequently from P3 we see that if $a_0 = 0$ in the continued fraction expansion of α then $\beta(\alpha)$ and the derived sequence $\beta'(\alpha)$ can never be identical.

Self Derived β - Sequences

If the β - sequence for α is identical to its derived sequence $\beta'(\alpha)$ we say that the sequence $\beta(\alpha)$ is self derived.

From above we have seen that $\beta(\sqrt{2})$ is self derived.

Theorem 2

$\beta(\alpha)$ is self derived if and only if, $\alpha = \alpha'$ and either

$$\alpha = \{n; n+1, n+1, \ldots\} = \frac{n - 1 + \sqrt{(n+1)^2 + 4}}{2},$$

in which case $\{\alpha\} < 1/2$ or

$$\alpha = \{n; 1, n, 1, n, 1, \ldots\} = \frac{n + \sqrt{n^2 + 4n}}{2},$$

in which case $\{\alpha\} > 1/2$, and n is the integer part of α.

Proof: (a) If $\beta(\alpha)$ is self derived, $\{\alpha\} < 1/2$ and $\alpha = \{a_0; a_1, a_2, a_3 \ldots\}$ then from Theorem 1,

$$\alpha' = 1/\{\alpha\} - 1 = \{a_1 - 1; a_2, a_3, \ldots\}, a_1 \geq 2 \text{ and } \alpha = \alpha'.$$

Consequently, matching terms in the continued fraction expansions of α and α' gives $a_0 = n, a_1 = a_2 = a_3 = \ldots = n + 1$ where n is a postitive integer.

Similarly if $\{\alpha\} > 1/2$ then $\alpha' = \{a_2; a_3, a_4, \ldots\}$ and $a_1 = 1$. So matching terms in the continued fraction expansions gives

$$a_0 = a_2 = a_4 = \ldots = n,$$

$$a_1 = a_3 = a_5 = \ldots = 1.$$

(b) If $\alpha = \{n; n+1, n+1, \ldots\}$ and $\{\alpha\} < 1/2$ then from Theorem 1

$$\alpha' = 1/\{\alpha\} - 1 = \{n; n+1, n+1, \ldots\} = \alpha \text{ and thus,}$$

$$\beta(\alpha) = \beta(\alpha') = \beta'(\alpha).$$

Similarly, if $\alpha = \{n; 1, n, 1, \ldots\}$ and $\{\alpha\} > 1/2$ then from Theorem 1

$$\alpha' = \{n; 1, n, 1, \ldots\} = \alpha \text{ and again, } \beta(\alpha) = \beta(\alpha') = \beta'(\alpha).$$

We note that although the terms in the derived sequence for α are independent of the value of $a_0 = [\alpha]$ we do need to know the value of a_0 to characterise a self derived sequence.

Examples of α which have self derived β - sequences are:

a)
$$n = 1, \alpha = \alpha' = \sqrt{2} = \{1; 2, 2, 2, \ldots\}\ ,\ \{\alpha\} = \sqrt{2} - 1 < 1/2$$
$$n = 2, \alpha = \alpha' = (1 + \sqrt{13})/2 = \{2; 3, 3, 3, \ldots\}\ ,\ \{\alpha\} = (\sqrt{13} - 3)/2 < 1/2$$
$$n = 3, \alpha = \alpha' = 1 + \sqrt{5} = \{3; 4, 4, 4, \ldots\}\ ,\ \{\alpha\} = \sqrt{5} - 2 < 1/2$$
$$n = 4, \alpha = \alpha' = (3 + \sqrt{29})/2 = \{4; 5, 5, 5, \ldots\}\ ,\ \{\alpha\} = (\sqrt{29} - 5)/2$$

b)
$$n = 1, \alpha = \alpha' = (\sqrt{5} + 1)/2 = \{1; 1, 1, 1, \ldots\}\ ,\ \{\alpha\} = (\sqrt{5} - 1)/2 > 1/2$$
$$n = 2, \alpha = \alpha' = (\sqrt{3} + 1) = \{2; 1, 2, 1, \ldots\}\ ,\ \{\alpha\} = (\sqrt{3} - 1) > 1/2$$
$$n = 3, \alpha = \alpha' = (\sqrt{21} + 3)/2 = \{3; 1, 3, 1, \ldots\}\ ,\ \{\alpha\} = (\sqrt{21} - 3)/2$$
$$n = 4, \alpha = \alpha' = 2 + 2\sqrt{2} = \{4; 1, 4, 1, \ldots\}\ ,\ \{\alpha\} = 2\sqrt{2} - 2 > 1/2$$

Corollary 3

If $\beta(\alpha_1)$ and $\beta(\alpha_2)$ are both self derived then,

a) $[\alpha_1'] \geq [\alpha_2'] \geq 1$ and $0 < \{\alpha_1'\} < \{\alpha_2'\} < 1/2$ whenever $0 < \{\alpha_1\} < \{\alpha_2\} < 1/2$.

b) $[\alpha_2'] \geq [\alpha_1'] \geq 1$ and $1/2 < \{\alpha_1'\} < \{\alpha_2'\} < 1$ whenever $1/2 < \{\alpha_1\} < \{\alpha_2\} < 1$.

Proof: The proof of both parts follows from Corollary 2 and the fact that

$$\alpha_1 = \alpha'_1, \alpha_2 = \alpha'_2$$

.

From Theorem 2 and the graph of α' against $\{\alpha\}$ it is seen that values of α for which $\beta(\alpha)$ is self derived are obtained from the graph by reading the value of α' at the appropriate point of intersection of the graph and straight lines of the form $\alpha' = n + \{\alpha\}$ where n is a positive integer.

4. Characteristics and the Rapid Generation of the β-Sequence

Consider the β - sequence with $\alpha = \tau - 1 = 0.618\ldots$ (see Section 1, example (f)) which is $1, 0, 1, 1, 0, 1, 0, 1, 1, 0, 1, 1, 0, 1, 0, \ldots$

It would be convenient to have a representation of the β - sequence which allowed us to indicate the number of terms in each string.

To do this we use a method based on Christoffel [2]. Firstly replace each zero by c and each unit by d. Hence, we represent the sequence by $dcd^2cdcd^2cd^2cd\ldots$ This we will call the characteristic of α. In general, the characteristic is obtained from a β - sequence by replacing a string of units of length k by d^k or a string of zeros of length j by c^j.

In a similar way, we see that if $\alpha = \sqrt{2} - 1 = \{0; 2, 2, 2, \ldots\}$ then from Section 1, example (h) the characteristic is $cdcdc^2dcdc^2dcdc\ldots$

From these examples we observe that if we start with $c\ d$ then we can obtain a longer portion of the characteristic and consequently the β - sequence by using information from the continued fraction expansion of α.

The following method is due to Markov [6] and is described in Venkov [9]. An alternative elegant procedure has been developed by Fraenkel et.al. [3].

In general to form the characteristic we form the following subsequences,

$$c_0 = c,\ c_1 = c^{a_1-1}d, \text{ where } \alpha = \{0; a_1, a_2, a_3, \ldots\}.$$

Then we form $c_{i+1} = (c_i)^{a_{i+1}-1}c_{i-1}c_i$ for $i \geq 1$.

Finally by joining (concatenating) our subsequences we form the characteristic $c_1c_2c_3c_4\ldots$.

This result will be formally proved in Theorem 3 and we now illustrate the procedure for,

a) $\alpha = \tau - 1 = 0.618\ldots = \{0; 1, 1, 1, \ldots\}.$

$$c_0 = c,\ c_1 = c^0 d = d,\ c_{j+1} = (c_j)^0 c_{j-1} c_j = c_{j-1} c_j;\ \ j \geq 1.$$

Thus $c_2 = c_0 c_1 = cd, c_3 = c_1 c_2 = dcd, c_4 = c_2 c_3 = cddcd$, and it is seen that $c_1 c_2 c_3 \ldots$ is indeed the characteristic.

b)
$$\alpha = \sqrt{2} - 1 = \{0; 2, 2, 2, \ldots\}.$$
$$c_0 = c,\ c_1 = cd,\ c_{j+1} = c_j c_{j-1} c_j;\ \ j \geq 1.$$

Thus $c_2 = c_1 c_0 c_1 = cdccd,\ \ c_3 = c_2 c_1 c_2 = cdccdcdcdccd$, and again the characteristic is generated as before.

We note that in both examples α is of the form $\{0; n, n, n, \ldots\}$ and in these circumstances $a_j = n$ for $j \geq 1$ in the general procedure described above.

We now proceed to develop Theorem 3.

Suppose $\alpha = \{a_0; a_1, a_2, \ldots\}$ then we define for $j \geq 0$,

$$\begin{aligned} t_j = \{a_j; a_{j+1}, a_{j+2}, \ldots\} &= a_j + \{0; a_{j+1}, a_{j+2}, \ldots\} \\ &= a_j + 1/t_{j+1} \end{aligned}$$

and $\quad f_j = 1/t_{j+1} = \{t_j\} = \{0; a_{j+1}, a_{j+2}, \ldots\} = t_j - a_j.$

In particular,

$$\begin{aligned} t_0 &= \alpha \ ,\ \ f_0 = \{\alpha\}, \\ t_1 &= 1/\{\alpha\} = \{a_1; a_2, a_3, \ldots\} = a_1 + 1/t_2 = a_1 + f_1, \\ f_1 &= 1/t_2 = t_1 - a_1 = \{t_1\} = \{1/\{\alpha\}\} = \{0; a_2, a_3, \ldots\}. \end{aligned}$$

If $0 < \alpha < 1$ then $\alpha = \{0; a_1, a_2, \ldots\}, t_1 = 1/\alpha$ and $f_1 = \{t_1\} = \{1/\alpha\}$.

Lemma 1.

If $0 < \alpha < 1$ and $m_k = [k/\alpha]$, where k is a positive integer, then

a) $\alpha m_k = \alpha(k a_1 + [k f_1]) = k - \alpha\{k f_1\}$

b) $[\alpha m_k] = k - 1\ \ ,\ \ [(m_k + 1)\alpha] = k$

c) $m_{k+1} - m_k = a_1 + \beta_k(f_1)$

d) $\beta_{m_k}(\alpha) = \beta_{m_{k+1}}(\alpha) = 1$, $\beta_j(\alpha) = 0$ for $j = m_k + 1, m_k + 2, \ldots, m_{k+1} - 1.$

Proof: (a) From our definitions,

$$\begin{aligned} m_k &= [kt_1] = [ka_1 + kf_1] = ka_1 + [kf_1] = ka_1 + kf_1 - \{kf_1\} \\ &= kt_1 - \{kf_1\} = k/\alpha - \{kf_1\} \end{aligned}$$

(b) Since $0 < \{kf_1\} < 1$ we see from (a) that $k - 1 < \alpha m_k < k$ and thus $[\alpha m_k] = k - 1$.

Now $\alpha(m_k + 1) = \alpha + \alpha m_k = k + \alpha(1 - \{kf_1\})$, from (a), and thus $k < \alpha(m_k + 1) < k + 1$, giving $[\alpha(m_k + 1)] = k$.

$$\text{(c)}\ \ \begin{aligned} m_{k+1} - m_k &= (k+1)a_1 - [(k+1)f_1] - ka_1 - [kf_1], \ \text{ from (a),} \\ &= a_1 + \beta_k(f_1). \end{aligned}$$

Since $0 < \{\alpha\} < 1$ for any irrational α we see that the results in Lemma 1 hold for any irrational α with α replaced by $\{\alpha\}$. This also follows from A23.

(d) $\beta_{m_k}(\alpha) = [(m_k + 1)\alpha] - [m_k\alpha] = k - (k - 1) = 1,$ from (b).

$\beta_{m_{k+1}}(\alpha) = [(m_{k+1} + 1)\alpha] - [m_{k+1}\alpha] = (k + 1) - k = 1,$ from (b).

Also

$$\sum_{j=m_k+1}^{m_{k+1}-1} \beta_j(k) = [m_{k+1}\alpha] - [(m_k + 1)\alpha] = 0, \ \text{from (b),}$$

and from P3 we have $\beta_j(\alpha) = 0$ for $j = m_k + 1, m_k + 2, \ldots, m_{k+1} - 1.$

Theorem 3.

The characteristic of α, $K(\alpha; c, d)$, is obtained from the β - sequence by firstly replacing the term $[\alpha]$ by c and $[\alpha]+1$ by d.

If we next define the subsequence $< c_k >$ by

$$c_o = c, \;\; c_1 = (c)^{a_1-1}d, \;\; c_{j+1} = (c_j)^{a_{j+1}-1}c_{j-1}c_j, \text{ for } j \geq 1$$

then $K(\alpha; c, d) =< c_k >= c_1 c_2 c_3 \ldots$.

Proof: We will prove the theorem for $0 < \alpha < 1$.

In the β sequence we select the string of terms, $\beta_{m_k}+1, \beta_{m_k}+2, \ldots, \beta_{m_{k+1}}$ which from Lemma 1 (c) will become $c^{m_{k+1}-m_k-1}d$ in $K(\alpha; c, d)$.

For the particular case where $k = 1,\ m_k = a_1$ and so beginning with the term $\beta_{a_1+1}(\alpha)$ in the β sequence we see that the corresponding string in the characteristic is,

$$(c)^{m_2-m_1-1}\, d\, (c)^{m_3-m_2-1}\, d\, (c)^{m_4-m_3-1}\, d \ldots$$

Since $[a_1\alpha] = 0$ and $[(a_1+1)\alpha] = 1$ from Lemma 1(b) we see that the string of terms $\beta_1(\alpha), \beta_2(\alpha), \ldots, \beta_{a_1}(\alpha)$ in the β - sequence becomes the string $(c)^{a_1-1}d$ in the characteristic.

Hence from Lemma 1 (c) we have

$$K(\alpha; c, d) = (c)^{a_1-1}\, d\, (c)^{a_1-1+h(1)}\, d\, (c)^{a_1-1+h(2)}\, d \ldots$$

where $h(k) = \beta_k(f_1)$.

Now since $0 < f_1 < 1$ the β - sequence for f_1 will consist of only zeros and units. Thus, if $K(f_1; c, d)$ is the characteristic of f_1 then the characteristic of α can be obtained from it by replacing each c in $K(f_1; c, d)$ by $c_1 = (c)^{a_1-1}d$ and each d by $d_1 = (c)^{a_1}d$ and adjoining c_1 on the left.

Thus

$$K(\alpha; c, d) = c_1 K(f_1; c_1, d_1)$$

where,

$$c_1 = (c)^{a_1-1}d \;\text{ and }\; d_1 = (c)^{a_1}d.$$

Similarly, we can show that

$$K(f_1; c_1, d_1) = c_2 K(f_2; c_2, d_2)$$

where,

$$c_2 = (c_1)^{a_2-1}d_1\ ,\ d_2 = (c_1)^{a_2}d_1, f_2 = 1/f_1 - a_2 \quad \text{and thus}$$

$$K(\alpha; c, d) = c_1 c_2 K(f_2; c_2, d_2).$$

Proceeding in this way we have,

$$K(\alpha; c, d) = c_1 c_2 \ldots c_j K(f_j; c_j, d_j)$$

where, for $j \geq 0$,

$$\begin{aligned} c_0 &= c, \ \ d_0 = d, \\ c_{j+1} &= (c_j)^{a_{j+1}-1} d_j, \\ d_{j+1} &= (c_j)^{a_{j+1}} d_j. \end{aligned}$$

We see that $d_{j+1} = c_j c_{j+1}$ and hence we have

$$c_0 = c, \ \ c_1 = (c)^{a_1-1}d, \ \ c_{j+1} = (c_j)^{a_{j+1}-1} c_{j-1} c_j, \ \ \text{for} \ \ j \geq 1.$$

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APPENDIX

RELATIONSHIPS FOR THE BRACKET FUNCTIONS $\{\ \}$ and $[\ ]$

We now consider some properties of the bracket functions of division namely $\{x\}$and$[x]$.

In what follows unless otherwise stated x and y are real, n is any integer $(\ldots-2,-1,0,1,2\ldots)$ k is a positive integer $(1,2,3,\ldots)$ and f is a real fraction such that $0 \leq f < 1$. Also $|x|$ is as usual the absolute value function defined as $|x| = x,$ if $x \geq 0$ and $-x$ if $x < 0$.

DEFINITIONS

Integer Function $[x]$

We define $[x]$ to be the largest integer not exceeding x. Hence

$$[x] = \text{ max integer } n : n \leq x$$

.

It follows that $[\pi] = 3$ but that $[-\pi] = -4$. Thus for non negative reals $[x]$ is the same as the truncation of x (that is x without the decimal fraction).

Fractional Part Function $\{\mathrm{x}\}$.

We define

$$\{x\} = x - [x]$$

As we will show below (A 12), $\{-\pi\} = 1 - \{\pi\} = .8584\ldots$

Note 1: If we consider a circle of unit circumference (not a unit diameter as is usual with complex algebra) then we can visualise $\{\pi\}$ as wrapping a string of length π clockwise around our circle from some point, which we call the origin, and ignoring full loops (of which there will be $[\pi]$): the length of the part of the string remaining is $\{\pi\}$.

For $-\pi$, we wrap our string anti clockwise. Again we ignore loops of which there will be $[\pi]$. But in this case $-[-\pi] =$ one more than the number of loops. And we emphasise that $\{-\pi\}$ is not equal to the length of the part of the string remaining but rather it is equal to the length of the arc of the circle not covered by this string. In each case the fractional part is equal to the anti-clockwise distance from the end of the string to the origin.

Note 2: These functions come from the **division algorithm**

$$y = ix + r, \quad \text{where } 0 \leq r < x$$

i, of course, is always an integer. If y and x are integers then r, the remainder, is also an integer. If both are rational then so is r. If at least one of x or y is irrational then so is r unless r is zero. It follows that $y/x = i + r/x$, and hence, for positive x and y, $i = [y/x]$ and $r/x = \{y/x\}$. Thus for x positive, it follows that i is identical to the integer part.

PROPOSITIONS

A1. $[n] = n$ Thus in particular $[-9] = -9$

A2. $x - 1 < [x] \leq x$.
The right inequality follows from the definition. The left inequality follows from the fact that there can be only one integer between x and $x - 1$ (this must be so otherwise this range would be greater than unity). Hence
$[19.6] = 19$ lies between 18.6 and 19.6.

A3. $[x] \leq x < [x] + 1$
This follows by rearranging A2.
Hence 19.6 lies between 18 and 19.

A4. $[n + f] = n$.
This follows from A1 and because $n \leq n + f < n + 1$
Thus $[19 + .6] = 19$

A5. $\{n + f\} = f$.
This follows immediately from the definition of $\{x\}$ and A4.
Thus $\{19 + .6\} = .6$

A6. $\{\{x\}\} = \{x\}$
Thus $\{\{19.6\}\} = \{19.6\}$

A7. $[n + x] = n + [x]$
As LHS $= [n + [x] + \{x\}] =$ RHS, from A4
Hence $[7 + 19.6] = 7 + [19.6]$

A8. $\{n + x\} = \{x\}$. This follows immediately from A7.
Hence $\{7 + 19.6\} = \{19.6\}$ and $\{-7 + 19.6\} = .6$. In particular
$\{-7 + .6\} = \{-6.4\} = .6$

A9. $[\,[y] + x] = [y] + [x] = [\,[x] + y]$, from A7
hence $[\,[19.6] + 3.5] = [19.6] + [3.5] = [\,[19.6] + 3.5]$

A10. $[- \mid x \mid] = -[\mid x \mid] - 1$, where x is not integer.
We note that $[- \mid n \mid] = -[n]$ follows from A1.
Now if $[\mid x \mid] = k$, then from A3, $k \leq \mid x \mid < k + 1$.

Hence $-k-1 < - \mid x \mid \leq -k$, from which the result follows.
For example $[\pi] = 3, [-\pi] = -4$. From this we immediately have

A11. $[n] + [-n] = 0$ and
$[x] + [-x] = -1$,**where x is not integer**.
Thus [-x] = - 1 -[x] = - (1 +[x]) hence $[-19.6] = -(1 + [19.6]) = -20$.

A12. $\{x\} + \{-x\} = 1$, where x is not an integer.
Follows by substituting in A11.
Hence $\{-x\} = 1 - \{x\}$ and thus $\{-\pi\} = 1 - \{\pi\} = .8584\ldots$
Another way of looking at this is that $\{-k+f\} = \{-(k-1)-(1-f)\} = f$
Hence $\{-19.6\} = \{-20 + .4\} = .4$, which agrees with A8 .

A13. $[x] + [y] \leq [x+y] \leq [x] + [y] + 1$
From A9 and A3, LHS = $[\ [x] + y] \leq [x+y] \leq x + y < [x] + [y] + 2 =$ RHS
Thus $[19.6] + [3.5] \leq [19.6 + 3.5] \leq [19.6] + [3.5] + 1$

A14. $[x/k]$ = number of positive integral multiples of k not exceeding x, where $x > 0$.
this follows directly from the division algorithm.

A15.

$$\begin{aligned}[kf] + [k(1-f)] &= k-1, \quad kf \text{ not an integer} \\ &= k, \quad kf \text{ an integer} \\ \text{where } 0 < \ & f \ < 1\end{aligned}$$

From A7 and A11.

$$\begin{aligned}[k(1-f)] &= k + [-kf] \\ &= k - 1 - [kf], \text{ In the case when } kf \text{ is not integer} \\ &= k - [kf], \text{ In the case when } kf \text{ is not integer}\end{aligned}$$

A16.

$$\begin{aligned}\{kf\} + \{k(1-f)\} &= 1, \quad kf \text{ not an integer} \\ &= 0, \quad kf \text{ an integer} \\ \text{where } 0 < \ & f \ < 1\end{aligned}$$

Follows as $\{k(1-f)\} = \{-kf\}$, from A8 and then use A12.

A17. $(n+1)/k \leq [n/k]+1$
$n = [n/k]k + r$, where r is an integer and $0 \leq r \leq k-1$
Hence $(n+1)/k = [n/k] + (r+1)/k \leq [n/k]+1$.

A18. $[\,[x]/k] = [x/k]$.
Let $i =$ LHS $= [\,[x]/k]$ then using A2, A3 and A17 we have
$i \leq [x]/k \leq x/k < ([x]+1)/k \leq i+1$
That is $i \leq x/k < i+1$, and the result follows from A3.

A19. The nearest integer to x is $k = [x+1/2]$.
this is equivalent to showing that
$|\,[x+1/2] - x| < 1/2$.
This follows from A2 since
$-1/2 = (x+1/2) - (x+1) < [x+1/2] - x \leq (x+1/2) - x = 1/2$
An alternative proof is the following: Firstly consider the case where x is nearer to $[x]$,
then $x = [x] + y$, where $y < 1/2$. In this case
$[x] < x + 1/2 = [x] + y + 1/2 < [x] + 1$ and hence $k = [x]$ as required.
Consider now the other possibility, that is x is nearer to $[x]+1$. In this case $x = [x] + 1 - y$ and it follows that $[x] + 2 > x + 1/2 > [x] + 1$ and thus $k = [x] + 1$, thus completing our proof.

A20. An immediate corollary of A19 is that if the integer $k = x \pm y$, where x, y are real such that $y < 1/2$ then $k = [x+1/2]$.
Thus for example, if τ is the golden section $(1.618\ldots)$ we can obtain the Fibonacci numbers directly from

$$F_k = [\tau^k/\sqrt{5} + 1/2]$$

This is less cumbersome than calculating the second term in the **Binet** expression $\sqrt{5}F_k = \tau^k - (-1/\tau)^k$

A20a. $-[-x]$ is the smallest integer not less than x.
Now $-x-1 < [-x] \leq -x$, from which we obtain the result by multiplying through by -1.
Thus $-[-19.6] = 20$ is the smallest integer greater than 19.

A21. As k increases without bound
$\lim_{[} k \to \infty]/k = x$.
Now by definition $kx - 1 < [kx] \leq kx$. Hence
$x - 1/k < [kx]/k \leq x$, and the result follows.
Thus $[1000 * 1.618]/1000$ is about 1.618.

A22. **a)** $[k\tau^2] = [\tau[k\tau]\,] + 1 = [k\tau] + k$
The equality of the first and third terms follows because $\tau^2 = \tau + 1$

Let $f = \{k\tau\} = k\tau - [k\tau]$
But $\tau^2 = 1 + \tau$ and hence
$k\tau^2 - [k\tau^2] = f$. That is $\{k\tau^2\} = \{k\tau\} = f$.

$$\begin{aligned}
\text{And as } 1/\tau &= \tau - 1, \\
-f/\tau &= f(1-\tau), \\
&= (k\tau^2 - [k\tau^2]) - (k\tau^2 - \tau[k\tau]\ ), \\
&= \tau[k\tau] - [k\tau^2].
\end{aligned}$$

The equality of the first and second terms follows as $0 < f < 1 < \tau$.
Thus with $k = 6$, we have $[\tau[6\tau]\ ] = 14$ and $[6\tau^2] = 15$.
b)$[\tau[k\tau]\ ] = [k\tau] + k - 1$.
Follows trivially from a).
c) $\{\tau[k\tau]\} = 1 - \{k\tau\}(\tau - 1)$.

$$\begin{aligned}
\{\tau[k\tau]\} &= \tau[k\tau] - [\tau[k\tau]\ ] \\
&= \tau[k\tau] - ([k\tau] + k - 1), \text{ from a)} \\
&= [k\tau](\tau - 1) - k + 1, \\
&= (k\tau - [k\tau])(\tau - 1) - k + 1, \\
&= k(\tau^2 - \tau - 1)] + \{k\tau\}(\tau - 1),
\end{aligned}$$

and the result follows as the first term in brackets is zero.

A23.

$$\begin{aligned}
[[k/\{\alpha\}]\{\alpha\}] &= k - 1, \text{ if } k \text{ is not a multiple of} \{\alpha\} \\
&= k, \text{ if } k \text{ is a multiple of} \{\alpha\}
\end{aligned}$$

To show this we consider $[k/\{\alpha\}] = k/\{\alpha\} - \{k/\{\alpha\}\}$
and thus $[k/\{\alpha\}]\{\alpha\} = k - \{k/\{\alpha\}\}\{\alpha\}$.
From this it is seen why we insisted that we use only a fractional part (or a value less than 1) as this ensures that the final product term must be less than unity. By taking the truncation the result follows.
Hence $[.3[15/.3]\ ] = 15$ but $[.31[15/.31]] = 14$.

A24. $\{\{kx\} - \{jx\}\} = \{(k-j)x\}$
From A8, LHS $= \{kx - [kx] - jx + [jx]\} = \{kx - jx\} =$ RHS
Thus $\{\{150 * 19.6\} - \{50 * 19.6\}\} = \{50 * 19.6\}$.

A25. $\{k\{x\}\} = \{kx\}$
The result follows from A8 and $k\{x\} = kx - k[x]$. Thus
$\{100\{19.6\}\} = \{100 * 19.6\}$

A26. Integer parts play an important role in the mathematics associated with the pigeon hole principle which can be stated as -

A26a. If $j + 1$ pigeons are placed into j holes then at least one hole will contain at least two pigeons. Thus if we try to place 11 pigeons into 10 holes then at least one hole must have two or more pigeons in it.
A generalisation of this is the following.

A26b. If k pigeons are placed into j holes than at least one hole will contain at least $h + 1$ pigeons where $h = [(k - 1)/j]$
To show this, consider what happens if the largest number of pigeons in a hole is h. Then it follows that the **total** number of pigeons cannot exceed jh. But from A2, $[(k-1)/j] \leq (k-1)/j$ and thus $kh \leq k-1$, hence it would be impossible for the total number of pigeons to add up to k and thus at least one hole must contain more pigeons than h.

Thus if we try to place $k = 23$ pigeons into $j = 10$ holes we have $h = [22/10] = 2$ and thus at least one hole must have at least 3 pigeons. To minimise the number of pigeons per hole we place 2 pigeons into each of the 10 holes. This leaves 3 over which forces at least one hole to have more than 2.
Note: With $k = j + 1$ we of course obtain A26a and with $k = ij + 1$ we are sure that there will be at least $i + 1$ pigeons in a hole.
Furthermore if we have a total of $j^2 + 1$ pigeons then there must be a least $j + 1$ pigeons in a hole. And this result enables us to prove the following nice result (Erdos 1935).
Given a sequence of exactly $j^2 + 1$ distinct integers either there is an increasing subsequence of $j + 1$ terms or a decreasing subsequence of $j + 1$ terms.