e the EXPONENTIAL - the Magic Number of GROWTH

Keith Tognetti
School of Mathematics and Applied Statistics
University of Wollongong NSW 2522 Australia

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Introduction

This Module is written as a self contained introduction to e, bringing together the main theorems and important properties of this fundamental constant of natural growth processes.

Not only is an axiomatic treatment given but this is complemented by some theorems that have been selected for their unexpected beauty.

It assumes only an elementary understanding of integration. Some anecdotes and history are also included. It will be seen that Euler's gamma constant also comes into the main theorem and this highlights the intimate interconnection between e, the area under the curve $\frac{1}{x}$, and the the truncated Harmonic series.

The number e is the base of Natural logarithms but it is not the base of Naperian logarithms as will be seen by glancing at the short historical note on Naperian logarithms in Appendix A.

e owes its importance to mathematics because we come across it in all natural growth processes.

e like π is transcendental and thus it cannot be expressed as the root of a finite polynomial equation with rational coefficients (go to Appendix C for some more about irrationals). If you read nothing else at least to help you get a feel for arithmetic look at Appendix D, the article by the Noble Prize winning physicist and sometimes lovable character Richard Feynman.

For a very general, exhaustive and more elementary introduction to e see Maor (References).

Let us begin our description of e by writing down the first digits

$$e = 2.718281828459045235360287 4...$$

We immediately note that 1828 is repeated at the beginning. This is rather exceptional as most of the important constants of mathematics that are irrational have no obvious pattern at their beginning. However it is almost certain that if we searched through the digits of most of these numbers we would come across any pattern we liked to define, as long as the pattern was not periodic. Thus for example the above list of digits will most certainly occur somewhere in the expansion of π; although be warned that it might take us a long time to come across them.

Our first introduction to the essential nature of e is through a problem in commerce -

Suppose that we have \$1 and invest it such that at the end of the first year we are given \$1 interest, thats right 100% interest. Now if instead of the full year we invested our money for $\frac{1}{10}$ year it would be only fair that we would get $\frac{1}{10}$ of this amount in interest, that is we would have a total amount of $\$1 + \frac{1}{10}$ at the end of this first period.

With simple interest we would get this same amount for each of our ten periods and hence the total interest would be \$1 which is the same as in the first case. This is the same as saying that our money has grown from \$1 to \$2 by an amount of $\$10 * \frac{1}{10}$.

Now if we compound our interest the following happens. At the end of the first $\frac{1}{10}$ of a year we would get $\$(1+\frac{1}{10})$ as before. However at the beginning of the second period we would invest this amount instead of \$1. It follows that at the end of the second period we would have $\$(1+\frac{1}{10})^2 = 1.21$ and so on until at the end of the year we would have $\$(1+\frac{1}{10})^{10} = \$2.5937$. Similarly at the end of the year with 100 periods we would have $\$(1+\frac{1}{100})^{100} = 2.7048..$ and for 1000 periods $\$2.71692..$

From this we find experimentally that as the number of periods increases the total amount appears to approach $\$e$. We note that the derivative of $(1+\frac{1}{n})^n$ is

$$(1+\frac{1}{n})^n\{\ln(1+\frac{1}{n}) - \frac{1}{(1+n)}\}$$

and that this is positive and decreases to zero as n increases. So this is further evidence that $(1+\frac{1}{n})^n$ converges as n becomes unbounded. However we require the Main Theorem to prove this conclusively.

You might like to check the following figures for reinvesting on a monthly (\$2.63), weekly (\$2.69) and daily (\$2.71) basis. Even if we reinvest at the end of a half year the amount is \$2.25 which is 12.5% better than simple interest.

This is one of the reasons why banks make such a profit from <u>simple</u> minded customers.

In general if instead of an increase of \$1 we had an increase of $\$r$ for the year (r is called the effective annual rate of interest) we can readily show in a similar way to the above that, with an initial investment of \$1, the amount at the end of the year is $(1+\frac{r}{n})^n$ and that this expression approaches e^r as n increases without bound (see A4, below).

For a rather dramatic example of the absurdity of believing that any economic system can really sustain exponential growth at even a “reasonable" growth rate see Appendix D.

Biological Growth Models

A similar exponential process occurs during the initial growth of many biological organisms. This occurs whenever the rate of change is proportional to the size or the number of the organisms. In this case we find that for example if y is the length of a root or the number of cells in an organism then

$$y = y(0)e^{rt}, \tag{1}$$

where $y(0)$ is the initial value , t is time and r is the intrinsic rate of increase.

This relationship is what is now generally referred to as Malthusian or exponential growth. However the original growth process referred to by Malthus was actually geometric. Knowing that $\frac{de^x}{dx} = e^x$, (from proposition Q4), it is seen that if we differentiate this equation on each side we get

$$\frac{dy}{dt} = y(0)re^{rt} = ry. \tag{2}$$

Hence

$$r = \frac{\frac{dy}{dt}}{y}. \tag{3}$$

Thus r, which we call the intrinsic rate of increase, is the actual rate of increase $\frac{dy}{dt}$ divided by the total length of, or number of, organisms.

We note that if y were say the number of individuals in a population then $\frac{\frac{dy}{dt}}{y}$, and hence r, is seen to be the average rate of growth per individual. It is a remarkable fact that, in a wide range of growth processes in nature and economics, this quantity is constant during the initial stages of such growth.

From (3) we see that

$$\ln\left\{\frac{y}{y(0)}\right\} = rt, \tag{4}$$

where from now on $\ln x$ means $\log_e x$.

(4) may be used as a rough check to test that we have exponential growth. We simply plot the logarithm of $\frac{y}{y(0)}$ against t. If this is a straight line then we have exponential growth.

This exponential growth is hard to deal with intuitively and in practice it is easier to use the doubling time t_d defined as the interval of time required for y to double. Then if we consider the population at any time t we will double our population at time $t + t_d$. Hence

$$\frac{y(t + t_d)}{y(t)} = 2.$$

Substituting in our exponential expression (1) we see that this reduces to $\exp(rt_d) = 2$, (that is the $y(t)$ cancels out and the expression is thus independent of t). From this we have

$$t_d = \ln\frac{2}{r}.$$

If r is measured as a % then this is equivalent to the following - the doubling time is roughly equal to 70 divided by the interest in % or

$$td \sim \frac{70}{r\%}.$$

This hyperbolic (or reciprocal) relationship is considered to be easier to grasp than the exponential. Thus for example with $r = 7$% the doubling time is 10yrs, and with $r = 10\%$ the doubling time is 7 yrs.

Other Properties and History

e has all sorts of other interesting properties. Perhaps the most beautiful relationship is Euler's formula which also involves π and i the square root of -1. This is

$$e^{i\pi} = -1.$$

Part of the beauty of this relationship lies in its unexpected simplicity. Here it is seen that i brings together e and π and unity, three of the fundamental constants of nature, together with the two operations minus and square root.

We will show in detail in Appendix A that Napier's logarithms are somewhat different to natural logarithms which use base e. However through formula AA there is a relationship but Napier was quite unaware of it so his work only implied the use of e.

It was not until Bernoulli (Johann I, 1704) that the symbol e and the name exponential were first used. This arose because it was necessary to have a way of representing e^x. In fact the term “Naperian antilogarithm" or e was not used until 1784.

The calculation of logarithms by infinite series was carried out by Gregory, Wallis and Halley (of Comet fame) beginning in the 1670’s. It was not until much later (1742) that William Jones (1675-1749) used exponents to calculate logarithms and allowed present day methods to be developed. However it should be emphasised that Euler was the first to introduce this concept, and the symbol e, for the base of natural logarithms in an unpublished paper of 1728 (he quoted e to 27 places).

The Layout of this Note

The formal part of this note begins with 2 definitions firstly of a function which is the area under the curve $\frac{1}{x}$ and then a definition for the inverse of this function.

This is followed by a set of propositions (P1, P2 etc) about the logarithmic function which are developed axiomatically from the definition. Then we present some properties of the exponential function e^x (Q1, Q2, Q3) developed axiomatically from its definition as an inverse of the exponential. Then we show that e^x is equal to its derivative (Q4). This is followed by some properties of the functions a^x and x^a (Q5, Q6).

Q7 gives a derivation for e which we used in the interest rate problem. Then we derive an expression for e^x as a series (Q8). From this we can show that e is irrational (see Appendix B). Then we give an independent proof of Q7 based on a note by Barnes which has the merit that it is coupled to Euler’s γ constant. At this stage we digress into Harmonic numbers.

Finally in contrast to the above axiomatic development we offer some alternative proofs A1, A2, ... A6. These have been selected simply on the basis of their elegance - and by elegance is meant unexpected simplicity.

An Axiomatic Approach

We begin with the definition

$$LN(x) = \int_1^x t^{-1} dt. \qquad \text{def 1}$$

It is seen that $LN(x)$ is simply a function of x at this stage. We hope to show that it does have the properties of a logarithm (see P4 below).

We also define the inverse of $LN(x)$ as $E(y)$

$$LN(x) = y$$

Hence

$$E(y) = LN^{-1}(y) = x. \qquad \text{def 2}$$

We hope to show that this inverse function is actually e^y (see Q3).

Properties of the logarithmic function $\ln x$

We now show that we can derive what we normally consider to be the properties of the logarithmic function from the above definition, def 1. We present these as a series of lemmas (minor theorems) or propositions - (it follows that a dillema is a theorem produced by a stupid person).

It is not until P4 that we are convinced that $LN(x)$ is in fact $\ln x$.

P1 $LN(1) = 0$, directly from def 1

P2 $\frac{dLN(x)}{dx} = \frac{1}{x}$.

This follows directly from def 1 by differentiation under the integral.

P3 The additive relation holds, $LN(ab) = LN(a) + LN(b)$.

Proof Let $z = LN(ax)$ then from the chain rule $\frac{dz}{dx} = \left(\frac{1}{ax}\right) a = \frac{1}{x}$.

However $\frac{d\,LN(x)}{dx} = \frac{1}{x}$ from P1. Hence as the functions $LN(ax)$ and $LN(x)$ have the same derivative they must differ only by a constant, that is

$$LN(ax) = LN(x) + c.$$

In particular this relationship holds for $x = 1$ and thus $c = LN(a)$, hence $LN(ax) = LN(x) + LN(a)$. With $x = b$ we have proved our lemma. •

P3a From P3 it follows directly that

$$\int_1^{xy} t^{-1} dt = \int_1^{x} t^{-1} dt + \int_1^{y} t^{-1} dt,$$

a result which is rather messy to prove directly by integration (is t^{-1} the only function that has this property ?) •

P4 $LN(a^n) = nLN(a)$.

This follows by continued application of P3. By substituting $\sqrt{n}a = y$ and thus considering y^n it follows that $\ln a^{\frac{1}{n}} = \frac{LN(a)}{a}$. From this it is easy to prove that $LN(a^r) = r\,LN(a)$, where r is rational. And hence as the irrationals are everywhere dense this result also holds for irrationals. •

Note at this stage we see that LN has all the properies of our familiar function $\log_e$.

P5 $\ln x$ is a convex increasing function without bound. It is increasing as its derivative is always positive. It is convex as its second derivative $-\frac{1}{x^2}$ is always negative.

It is unbounded as we can always obtain values of $\ln x > Q$ where Q is any large quantity we like to nominate. This is neatly demonstrated as follows. In $3^n = n \ln 3$. But $\ln 3 > 1$ and thus if $x = 3^n$ then $\ln x > n$, no matter how large n. •

P6

$$\int \ln x\, dx = x \ln x - x$$

This follows by integration by parts. •

P7 For $|x| \leq 1$, it can be shown from Taylor's formula that

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \ldots \quad \bullet\bullet$$

Properties of the Exponential Function e^y

We now define the inverse of the logarithm of x based simply on the def 1. Again as we do not know as yet that this is really e^y so let us call it $E(x) = y$, where $x = \ln y$.

Q1 $E(0) = 1$

this is merely acknowledging that $\ln 1 = 0$.

Q2 $E(u)E(v) = E(u+v)$

Let $E(u) = a$, $E(v) = b$, $E(u+v) = c$ then $u = \ln a$, $v = \ln b$, $u + v = \ln c$. It thus follows that $u + v = \ln ab$ is equivalent to $ab = c$ as required.

Definition We now define e as $E(1)$. def 3.

This is reasonable as if e is the base of our natural logarithm then we must have $\ln e = 1$. Hence

Q3 $E(n) = e^n$.

This follows immediately by the continued application of Q2, with $E(u) = E(v) = E(n)$.

In a similar way to P4 we are able to then show that

Q3a $E(r) = e^r$, where r is rational.

Q3b $E(x) = e^x$, where x is real. Suppose that x is irrational. Then for example by using continued fractions we can express x as the limit of a sequence of rationals. Thus the relationship holds for any rational as close to x as we like and thus in the limit it must hold for x irrational.

Thus we can now for the first time be convinced that if $x = \ln y$, then $y = e^x$.

Q4 $\frac{de^x}{dx} = e^x$.

From P2, $\frac{d \ln y}{dy} = \frac{dx}{dy} = \frac{1}{y}$. Hence as $\frac{dx}{dy}$ is always positive we can obtain the derivative of the inverse as

$$\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}} = y = e^x.$$

Note For any function $y = f(x)$ and its inverse $f^{-1}(y) = x$ we can see that $f(x)$ is symmetric to $f^{-1}(x)$ about the line $y = x$.

This is equivalent to showing that the point $P = (x, y)$ and the point $Q = (y, x)$ are symmetric about the line $y = x$ (which has slope 1). Now the slope of the line PQ is immediately seen to be -1. Also $OP = OQ = \sqrt{x^2 + y^2}$. Hence the result follows. Thus the graph of e^x is simply the graph of $\ln x$ reflected about the line $y = x$. Thus for example we see that the slope of e^x at $x = 0$ is the same as the slope of $\ln x$ at $x = 1$, namely unity.

Q5 The functions a^x and x^a.

Let $\ln e^{f(x)} = g(x)$, then by (Q3b) $e^{f(x)} = e^{g(x)}$ and thus $f(x) = g(x)$. Similarly if $h(x) = e^{\ln f(x)}$, taking logs we have $\ln h(x) = \ln f(x)$, and again $f(x) = h(x)$. Combining these we see that

$$f(x) = \ln e^{f(x)} = e^{\ln f(x)}.$$

From this we see that we can always write a as power of e or as a logarithm

$$a = e^{\ln a} = \ln e^a.$$

We can now write our functions very neatly as

$$y = a^x = e^{x \ln a} \quad \text{and} \quad x^a = e^{a \ln x}.$$

Also more directly from the series for e (Q8)

$$y = e^{x \ln 2} \sim 1 + x \ln 2 \quad \bullet$$

Q6 $\frac{d(a^x)}{dx} = a^x \ln a$ and $\frac{d(x^a)}{dx} = ax^{a-1}$.

Q7 $e^z = \lim_{n\to\infty} \left(1 + \frac{z}{n}\right)^n$

If $f(x) = \ln x$, then

$$\begin{aligned} f'(x) &= \tfrac{1}{x} = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}, \\ &= \lim_{h\to 0} \frac{1}{h}\{\ln(x+h) - \ln(x)\}, \\ &= \lim_{h\to 0} \frac{1}{h} \ln(1 + \frac{h}{x}). \end{aligned}$$

With $\frac{1}{x} = z$

$$\lim_{h\to 0} \frac{1}{h} \ln(1 + hz) = z.$$

Consequently

$$\begin{aligned} e^z &= \lim_{h\to 0} \exp[\ln(1+hz)^{1/h}] = \lim_{h\to 0}(1+hz)^{1/h} \\ &= \lim_{n\to\infty} (1 + \frac{z}{n})^n, \quad \text{if} \quad h = \frac{1}{n}. \end{aligned}$$

In particular with $z = 1$ we have independently shown that

$$e = \lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n \quad \bullet$$

Q8 $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots = \Sigma \frac{x^j}{j!}$, for $j \geq 0$.

This follows directly from the Taylor series expansion about $x = 0$ for $f(x) = e^x$. From this we have for $x = 1$

$$e = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \ldots = \Sigma \frac{1}{j!}, \quad \text{for} \quad j \geq 0,$$

which has reasonably fast convergence and so may be used to calculate e •

An example Suppose we seek an approximation to $y = 2^x$ for small x. From the binomial theorem

$$\begin{aligned} y &= (1+1)^x = 1 + x + \tfrac{x(x-1)}{2!} + \tfrac{x(x-1)(x-2)}{3!} + \tfrac{x(x-1)(x-2)(x-4)}{4!} + \ldots \\ &= 1 + x(1 - \tfrac{1}{2} + \tfrac{1}{3} - \tfrac{1}{4} + \ldots) + x^2(\tfrac{1}{2}\ldots) \\ &\sim 1 + x\ln 2. \end{aligned}$$

Q9 With Q8 as our definition for e we can now give the

Proof that e is irrational. Assume the opposite that is $e = \frac{p}{q}$ where p and q are integers. Then

$$\begin{aligned} \tfrac{p}{q} &= 1 + 1 + \tfrac{1}{2!} + \ldots + \tfrac{1}{q!} + \tfrac{1}{(q+1)!} + \ldots \\ (q-1)!p &= \left\{q! + q! + \tfrac{q!}{2!} + \tfrac{q!}{3!} + \ldots \tfrac{q!}{q!}\right\} + \tfrac{1}{(q+1)} + \tfrac{1}{(q+1)(q+2)} + \ldots \end{aligned}$$

But the expressions on the left hand side and in brackets are obviously integers, whereas the remainder

$$\frac{1}{(q+1)} + \frac{1}{(q+1)(q+2)} + \ldots < \frac{1}{(q+1)} + \frac{1}{(q+1)^2} + \ldots = \frac{\frac{1}{(q+1)}}{1 - \frac{1}{(q+1)}} = \frac{1}{q} \le 1.$$

Thus for our original assumption to be valid we require an integer to be equal to an integer plus a number less than one. As this is absurd our original asssumption is invalid and e must be irrational. •

Theorem M To show that

$$e = \lim_{n\to\infty}\left(1 + \frac{1}{n}\right)^n.$$

This presentation is based on Barnes, C. (1984). As a by-product it establishes the expression for Eulers constant γ, which is defined below. Alternative proofs are given in Q7, and A1.

Proof From P6

$$\int_{\frac{1}{(n+1)}}^{\frac{1}{n}} \ln x \, dx = \frac{1}{n(n+1)}\left[\ln\frac{(n+1)^n}{n+1} - 1\right]. \qquad (M1)$$

Also we know from the mean value theorem that there exists a c_n somewhere in the range of integration (that is $\frac{1}{(1+n)} < cn < \frac{1}{n}$) where

$$\int_{\frac{1}{(n+1)}}^{\frac{1}{n}} \ln x \, dx = \left(\frac{1}{n} - \frac{1}{(1+n)}\right) \ln c_n. \qquad (M2)$$

From (M1) and (M2)

$$\ln\left[\frac{(n+1)^n}{n^{n+1}}\right] - 1 = \ln c_n,$$

from which

$$\ln\left[\frac{\frac{(n+1)^n}{n^{n+1}}}{c_n}\right] = 1.$$

Hence

$$e = \frac{\frac{(n+1)^n}{n^{n+1}}}{c_n},$$

or

$$e = \left(1 + \frac{1}{n}\right)^n 1/nc_n. \qquad (M3)$$

Now let $a_n = \frac{1}{(nc_n)}$. Then because of the range of c_n we have

$$1 < a_n < 1 + \frac{1}{n}. \qquad (M4)$$

Hence we have the remarkable result that for all positive n

$$e = a_n \left(1 + \frac{1}{n}\right)^n. \qquad (M5)$$

As $a_n > 1$ it follows immediately from (M5) that for example

$$e > \left(1 + \frac{1}{2}\right)^2 = 2\frac{1}{4}.$$

Also because a_n is in the above sandwich (M4), in the limit as $n \to \infty$, $a_n = 1$ and thus our theorem is proved. •

Definition We define Euler's constant as

$$\gamma = \lim_{n\to\infty}\left[\left\{1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n}\right\} - \ln(n+1)\right].$$

Corollary M

$$\gamma < \frac{\pi^2}{6} = 1.64..$$

Proof

From (M5)

$$e = a_n \left(1 + \frac{1}{n}\right)^n = \frac{a_n(n+1)^n}{n^n}.$$

Thus taking logarithms

$$1 = \ln a_n + n\{\ln(n+1) - \ln n\},$$

or

$$\frac{1}{n} = \ln a_n^{1/n} + \ln(n+1) - \ln n.$$

From this by substituting successive values 1, 2, 3, ... for n, we obtain the equations

$$1 = \ln a_1 + \ln 2 - \ln 1$$

$$\frac{1}{2} = \ln a_2^{1/2} + \ln 3 - \ln 2$$

$$\frac{1}{3} = \ln a_3^{1/3} + \ln 4 - \ln 3$$

....

$$\frac{1}{n} = \ln a_n^{1/n} + \ln(n+1) - \ln n.$$

Summing

$$1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n} - \ln(n+1) = \ln a_1 a_2^{1/2} a_3^{1/3} \ldots a_n^{1/n}. \quad (M6)$$

We note that the continued product on the right hand side is increasing in n. Also from (M4) $a_n < 1 + \frac{1}{n}$, hence

$$\ln a_1 a_2^{1/2} a_3^{1/3} \ldots a_n^{1/n} = \ln a_1 + \frac{1}{2} \ln a_2 + \ldots \frac{1}{n} \ln a_n,$$

$$< \ln\left(1+\frac{1}{1}\right)+\frac{1}{2}\ln\left(1+\frac{1}{2}\right)+\ldots+\frac{1}{n}\ln\left(1+\frac{1}{n}\right).$$

Furthermore from (M4), $a_n > 1$. Thus from (M5), $1 < \left(1+\frac{1}{n}\right)n < e$. Hence $\ln 1 < \ln\left(1+\frac{1}{n}\right)^n < \ln e$, or $0 < \ln\left(1+\frac{1}{n}\right) < 1/n$. Substituting this into the inequality

$$\ln a_1 a_2^{1/2} a_3^{1/3} \ldots a_n^{1/n} < 1 + \frac{1}{2^2} + \frac{1}{3^2} + \ldots \frac{1}{n^2}.$$

But it can be easily shown that the infinite series $1 + \frac{1}{2^2} + \frac{1}{3^2} + \ldots$ is convergent. In fact it has the value $\frac{\pi^2}{6} \cong 1.644$ (see Knuth - would you believe that if we select two integers at random the probability that they don't have a common factor is $\frac{6}{\pi^2}$). Thus as the continued product is bounded, γ the l.h.s. of (M6) is bounded and hence $\gamma < 1.64\ldots$ •

Note - It can be shown numerically that $\gamma = 0.5772\ldots$.

Euler's constant features widely in many diverse areas of mathematics such as number theory and modified Bessel functions. Also Knuth (1973) (with a rather more introductory account of this in Graham, Knuth and Patashnik) gives an interesting treatment of the above series for the summation of the first n reciprocals which he defines as

$$H_n = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots + \frac{1}{n} = \Sigma\frac{1}{k}, \quad k \leq n,$$

where H stands for harmonic number. Hence the infinite series H_∞ is the harmonic series.

It follows that for large values of n, $H(n) - \gamma$ approaches $\ln n$. It should be noted that it is only for very large values that the term γ can be discarded - for example it takes more than 10^{43} terms for $H(n)$ to exceed 100.

A more robust formulation which is a reasonable approximation for smaller values of n is

$$\lim_{n\to\infty} H(n) = \ln n + \gamma + \frac{1}{2n} - \frac{1}{12n^2} + \frac{1}{120n^4} - s,$$

where $0 < s < \frac{1}{256n^6}$ and

A Connection Between the Harmonic Number and the Area under the curve $y = \frac{1}{x}$, for $x > 1$ can be developed from first principles as follows. By simple graphical construction the area under this curve, for values of x between $x = 1$ and $x = n$ is less than the sum of the n rectangles constructed, firstly on the base $x = 1$ to 2 and of height $1(= \frac{1}{1})$ and then on the base $x = 2$ to 3 and of height $\frac{1}{2}$ and so on until we construct the last rectangle on the base $x = n$ to $n + 1$ of height $\frac{1}{n}$. That is $\ln n < H_n$.

Similarly the area under this curve is greater than the sum of the of the $n - 1$ rectangles constructed firstly on the base $x = 1$ to 2 and of height $\frac{1}{2}$ and then on the base $x = 2$ to 3 and of height $\frac{1}{3}$ and so on until we construct the last rectangle of height $\frac{1}{n}$ on the base $x = n - 1$ to n. That is $\ln n > H_n - 1$.

This gives us the intuitive basis for the entry into our axiomatic treatment beginning with def 1.

Now we have simply shown that $1 + \ln n > H_n$ and this gives a better upper bound, that is $\gamma < 1$ than Corollary M. So why have we bothered with the corollary? Well it does give some other rich relationships and it connects nicely with our main theorem. So it just could be valuable at a some later stage and that is what the essence of creative mathematics is all about - stumbling upon unexpected connections with other areas.

A comment : The Generalised Harmonic Number

$$H_n^{(r)} = 1 + \frac{1}{2^r} + \frac{1}{3^r} + \frac{1}{4^r} + ... + \frac{1}{n^r} = \Sigma \frac{1}{k^r}, \quad k \leq n.$$

For any value of $r > 1$, $H_\infty(r)$ is convergent. H_∞ is just convergent in the sense that if r is just a touch greater than 1 the series converges. Hence we woud expect from this that H_n increases very slowly with n as we have found out above.

$H_\infty(r)$ is the corresponding infinite series is the Rieman Zeta function and is usually represented by $\zeta(r)$. Now if we attempt to solve $\zeta(r) = H_\infty^{(r)} = 1$ for r, it is seen that the roots must be complex. The Riemann hypothesis makes the remarkable claim that all such roots lie along the one vertical straight line in the complex plane such that their real parts are equal to $\frac{1}{2}$.

Alternative Proofs

With such an important constant as e one finds that there are many rich relationships amongst the various associated expressions. Consequently it would not be expected that any particular axiomatic development would present the most elegant proofs. So we include some of these.

A1 To prove $e = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n$

Let $y = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n$ then

$$\begin{aligned}\ln y &= \lim_{n\to\infty} n\ln\left(1+\frac{1}{n}\right)\\ &= \lim_{n\to\infty}\frac{\ln\left(1+\frac{1}{n}\right)}{\frac{1}{n}}.\end{aligned}$$

As this is in the form 0/0 we use l'Hospitals rule and differentiate

$$\begin{aligned}\ln y &= \lim_{n\to\infty}\frac{-\frac{1}{n^2}\left(1+\frac{1}{n}\right)}{-\frac{1}{n^2}},\\ &= 1,\end{aligned}$$

and this shows that $y = e$ as required.

Notice that we have assumed P2, that is that we know the derivative of a logarithm.•

A2 Let f be a function such that for every pair of reals x, y

$$f(x+y) = f(x)f(y). \qquad (M).$$

We will say that f satisfies the multiplicative property.

Then we require to prove the following:

if $f(x)$ is continuous and is not identically zero, and $f(1) = a > 0$, and furthermore $f(x)$ satisfies the multiplicative property then

$$f(x) = a^x \quad \text{for all} \quad x.$$

Proof Now $f(x) = f\{(x-1)+1\} = f(x-1)f(1)$.

Hence if $f(1) = 0$ then $f(x) = 0$ for all x. Hence we assume the opposite.

Suppose $f(1) = a > 0$. Then again

$$\begin{aligned} f(x) &= f(x-1)f(1) = f(x-1)a = f(x-2)a^2 = f\{x-(x-1)\}a^{x-1} \\ &= f(1)a^{x-1} = a^x. \end{aligned}$$

Note that this shows that (M) holds only when x is integer as we subtracted integers from x to obtain 1. To prove this for x rational say $\frac{m}{p}$ where m and p are integers we firstly write

$1 = \frac{1}{p} + \frac{1}{p} + \ldots \frac{1}{p}$, that is p terms.

Thus applying (M) repeatedly

$$f(1) = \left\{ f\left(\frac{1}{p}\right) \right\}^{1/p}.$$

Also

$$\begin{aligned} f\left(\tfrac{m}{p}\right) &= f\left(\tfrac{1}{p} + \tfrac{1}{p} + \ldots \tfrac{1}{p}\right), \text{ where we now have } m \text{ terms} \\ &= \left\{ f\left(\tfrac{1}{p}\right) \right\}^m = \{f(1)\}^{m/p} = a^{m/p} \end{aligned}$$

or $f(x) = a^x$, as required for x rational. It follows that as the rationals are everywhere dense that this result also holds for real functions provided that $f(x)$ is continuous. •

A3 We wish to show that the solution to

$\frac{dy}{dx} = y$ where $y(0) = 1$,

has the multiplicative property of A2.

That is we claim that $y(x+v) = y(x)y(v)$.

Proof Let $z(x) = y(a+x)$. Then $z'(x) = z(x)$.

Now $\frac{d}{dx}\frac{y(x)}{z(x)} = \frac{zy' - z'y}{z^2} = \frac{zy - zy}{z^2} = 0$.

Hence $\frac{y(x)}{z(x)} = K$, a constant, or

$$y(x) = Kz(x) = K * y(a + x).$$

Hence as $y(0) = 1$, with $x = 0$ we have $K = \frac{1}{y(a)}$. From this

$$y(a + x) = y(a) * y(x),$$

as required.

We also know independently that $y(x)$ has a unique solution and hence the multiplicative property is unique. •

A4 Assuming A1, $e = \lim_{t\to\infty}\left(1 + \frac{1}{t}\right)^t$, we wish to show that

$$e^x = \lim_{n\to\infty}\left(1 + \frac{x}{n}\right)^n.$$

We note firstly that t may be real, that is it does not have to range over the integers for A1 to be valid.

Now

$$\left(1 + \frac{x}{n}\right)^n = \left\{\left(1 + \frac{x}{n}\right)^{n/x}\right\}^x = \left\{\left(1 + \frac{1}{t}\right)^t\right\}^x$$

with $t = \frac{n}{x}$. The result then follows by taking the limit. •

A5 Let $f(x) = b^x$

Then from Q6

$f'(x) = b^x \ln b$ and thus $f'(0) = \ln b$

But from first principles

$$f'(x) = \lim_{h\to 0} \frac{b^{x+h} - b^x}{h}.$$

Thus with $x = 0$

$$\ln b = \lim_{h\to 0} \frac{b^h - 1}{h}. \quad \bullet$$

The relationship A5 automatically resolves the apparent conflict with the expression

$$\int_a^b x^u dx = \frac{b^{u+1} - a^{u+1}}{(u+1)},$$

when $u \to -1$, say $-1+h$, and with $a = 1$, it is seen from the above limit that this approaches $\ln b$ as required when $h \to 0$ •

A6 Using Q8 to define e^x we wish to establish the multiplicative property.

Now

$$e^u e^v = \sum_{m=0}^{\infty} \frac{u^m}{m!} \sum_{m=0}^{\infty} \frac{v^m}{m!}$$

But it is known that (the Cauchy convolution form for the product of two series)

$$\sum_{n=0}^{\infty} a_n \sum_{n=0}^{\infty} b_n = \sum_{n=0}^{\infty} c_n$$

where $c_n = \sum_{k=0}^{n} a_{n-k} b_k$.

Hence

$$e^u e^v = \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{u^{n-k}}{(n-k)!} \frac{v^k}{k!},$$

and by the binomial expansion

$$(u+v)^n = \sum_{k=0}^{n} n! u^{n-k} \frac{v^k}{(n-k)!k!},$$

from which

$$e^u e^v = \frac{\sum_{n=0}^{\infty} (u+v)^n}{n!} = e^{u+v}.$$

References

Barnes, C. "Eulers Constant and e", Am Math Monthly Aug 84, p428.

Coolidge, J.L. "The Mathematics of Great Amateurs", (Oxford, 1949).

Eves, H. “Great Moments in Mathematics", (Mathematical Association of America).

Knuth, D.E. “The Art of Computer Programming", (Section 1.2.7 Harmonic Numbers p73) vol 1 (2nd edition) 1973.

Graham, Knuth, D. and Patashnik, O. “Concrete Maths" - (Addison Wesley - 2nd Edition 1994) - Concrete is short for continuous plus discrete).

Maor, Eli. “e: The Story of a Number" (Princeton University Press, 1994).

Struik, D.J. “A Source Book in Mathematics 1200-1800", (Harvard, 1969).

APPENDIX A

Napier (1550-1617), a Scottish Laird, was one of the great amateurs of mathematics. He began to develop logarithms in 1594 and worked on this task until his death. Amongst other things he was obsessed with demonstrating that the then reigning pope was the anti christ. In fact Napier believed that his now forgotten work on the Apocalypse of St John was his greatest service to mankind and this belief was bolstered by the fact that his book on this subject went through 20 editions. Newton had a similar obsession with the Apocalypse which might indicate (at least to a statistician) that if we started off budding young geniuses on this study it could lead to great mathematical breakthroughs.

Briggs was the first occupant of a chair of mathematics in Britain. This was in Geometry at Gresham College in London. Later he was the first to occupy the Savillian chair of geometry at Oxford. He extended Napier’s work to base 10 and went to visit him after hearing of his wonderful discovery and communicating with him. On the interminable journey by coach to Edinburgh Briggs recorded some of his thoughts in his diary - how tall a forehead must his nobleman possess in order to house the brains that have discovered so remarkable an invention and so on. His arrival in Edinburgh was very much delayed and Napier confessed to a friend “Ah John, the Professor will not come".

At that very moment a knock was heard at the gate and the Professor was ushered into the nobleman's presence. For almost a quarter of an hour each man beheld the other without speaking a word. Then Briggs said "My Lord, I have undertaken this long journey purposely to see your person, and to learn by what engine of wit or ingenuity you came first to think of this most excellent help in astronomy. But, my lord being by you found out, I wonder nobody found it out before when now known it appears so easy."

In understanding the importance the work of Napier it should be realised that the mathematics of his day was very deficient in both concepts and notation. Not only was there no concept of logarithm but the concept of powers was only emerging and for example the symbol x^n as a shorthand for $x * x * \ldots$ had not been introduced. It is of interest to note that only because of the authority of Napier's work did the decimal notation became generally accepted, thus = 2.083 instead of the clumsy $2\frac{83}{1000}$.

The description of the technique he used for calculating his "logarithms" and tables of such "logarithms" are best described in his second book which was first published in Latin in 1619. This was translated from the Latin into English and published by Blackwood, Edinburgh in 1889, with the title "The Construction of the Wonderful Canon of Logarithms".

Curiously enough although the term logarithm occurs in the title the text uses only the term *artificial number*. As we will show shortly his logarithm is not what we now know as the natural logarithm. Detailed accounts of his work are contained in Struik, D.J. (1969) and Coolidge, J.L. (1949). Unfortunately his presentation is almost unintelligible to a present day mathematician and although his work is one of the great moments in mathematics it is in effect an ugly dead end, similar to the dead end taken in the pursuit of the Ptolemaic system of epicycles as a description of planetary motion.

The basis for his idea was probably seen in the formula

$$\sin A \sin B = \frac{1}{2}\{\cos(A-B) - \cos(A+B)\},$$

which was certainly known in Napier's time.

Now Stifel had earlier in “Arithmetica Integra" (1543) recorded the first observation that the terms of the geometric progression 1, r, r^2, $\ldots$ correspond to the arithmetic progression $0,\ 1,\ 2,\ \ldots$. Stifel also noted that multiplication of two terms in this g.p. yields a term whose exponent is in effect the sum of the corresponding two terms in the a.p. Similarly he implied that division was coupled with subtraction of exponents and he was even able to manipulate negative and fractional exponents. All this was recorded without any special mathematical notation. It appears that Napier did know of this work but, incredibly, failed to see the relevance of exponents on his logarithms. If he had picked it up it would most surely have accelerated the development of mathematics.

Because of his interest in spherical trigonometry he based his work on the logarithms of sines however let us warn you now that his use of the terms radius and sine simply referred to intervals on a straight line (AZ and DZ in the following). In effect he said the following. Lay out two lines AZ and $A'Z'$. Measuring in the direction from Z to A, the point A corresponds to what Napier calls the sine of 90 degrees which he has approximated by 10^7 as he could not of course represent infinity. Along $A'Z'$ he lays out the corresponding “logarithms" of these sines, thus he has A' representing 0, or in his notation, “logarithm" $10^7 = 0$. Now he lays out points B, C, D, E, along AZ and, the points B', C', D', E', $\ldots$ he lays out along $A'Z'$. It will be shown that $A'B'$ corresponds to his logarithm of AZ, $A'C'$ corresponds to his logarithm of BZ and so on.

Now he has a point initially at A moving along AZ, with a velocity proportional to its distance from Z and this corresponds to a point moving along $A'Z'$ with constant velocity equal to the initial velocity of the other point when it is at A. Suppose that on AZ each of the intervals AB, BC, $CD \ldots$ are traversed in the same small interval t.

Napier assumed that if t is small enough then the velocity of the point throughout the interval is constant and equal to the velocity at the beginning of the interval. From this the intervals AZ, BZ, CZ, $\ldots$ form a decreasing g.p. as we will now demonstrate. Let the velocity of the point throughout the interval whose initial distance is y from B be v, in the direction AB, such that $v = ky$. Thus for example

$$DZ = CZ - CD.$$

The velocity of the point at C is $k(CZ)$. This is now assumed constant over the interval CD. Hence $CD = k(CZ)t$. From which

$$DZ = CZ - k(CZ)t = CZ(1 - kt).$$

From this it is seen that any length is a constant times the preceeding length and our result follows. Initially Napier took kt to be $\frac{1}{10^7}$. He took the distances $A'B'$, $A'C'$, ... to be 1,2,3, ..., however any arbitrary increasing a.p. would be just as satisfactory.

Translating the above into present day calculus, which was of course unavailable to Napier, we consider the motion of two particles the first which we will call P, travelling along AB. The second particle P' (the image of P) we will consider travelling along $A'B'$. Let $B = a(= 10^7)$, $PZ = y$ and let $AP' = x$. Suppose the first particle is at point A initially (at time $t = 0$) that is $y(0) = a$. Then its velocity at this point is ka. Napier then assumed that this is the same as the constant velocity of the point P' moving along the line $A'Z'$. From this the velocity of P is $\frac{d}{dt}(a - y) = -\frac{dy}{dt} = ky$. From this we know that $y = ae^{-kt}$, and that $kt = \ln \frac{a}{y}$ where $\ln$ is the natural log to base e. Now if $A'C' = x$, then as the point moves along the line $A'Z'$ with the constant velocity a, then $x = kat$.

Napier defined this to be the Naperian logarithm which we will call $\mathbf{Nap\ y}$. Thus we have

$$\begin{aligned} \mathbf{Nap\ y} = \mathbf{a} \ln \frac{\mathbf{a}}{\mathbf{y}} &= a \ln a - a \ln y \\ &= \ln \left(\frac{a}{y}\right)^a = a \log_{1/e} \frac{y}{a}. \quad (AA) \end{aligned}$$

In the particular case where $y = a$, $\mathrm{Nap}\ a = 0$ which satisfies Napier's requirement that the point A correspond to log zero. Of course if $a = 1$, then $Nap\ y$ is simply

$$-\ln y \ \text{ or } \ \log_{1/e} y.$$

It should be emphasised that Napier's logarithm Nap is actually the natural logarithm of $\left(\frac{a}{y}\right)^a$ and thus it is inadvisable to couple the two names together.

We note that

$\mathrm{Nap}\ zy = a \ln \frac{a}{zy} = a(\ln a - \ln z - \ln y) = a \ln a - a(\ln z + \ln y),$

$\mathrm{Nap}\ \frac{z}{y} = a \ln \frac{ay}{z} = a(\ln a + \ln y - \ln z)$

$\mathrm{Nap}\ z + \mathrm{Nap}\ y = 2a \ln a - a(\ln z + \ln y) = 2a \ln a - a \ln(z + y),$

$\mathrm{Nap}\ z - \mathrm{Nap}\ y = a(\ln y - \ln z)$

and thus

$\mathrm{Nap}\ zy = (\mathrm{Nap}\ z + \mathrm{Nap}\ y) - a \ln a.$

$\mathrm{Nap}\ \frac{z}{y} = (\mathrm{Nap}\ y - \mathrm{Nap}\ z) + a \ln a.$

Consequently Nap does satisfy the additive rule directly if we subtract a constant. Napier also noted that if $\frac{a}{b} = \frac{c}{d}$ then

$\mathrm{Nap}\ a - \mathrm{Nap}\ b = \mathrm{Nap}\ c - \mathrm{Nap}\ d.$

Napiers logarithms were certainly used widely throughout Europe and this made Napier very famous. Whether he was renumerated for his great achievments is unknown. Laplace summed up their impact by saying that they “by shortening the labours of an astronomer double his life".

Briggs was very much obliged to Napier for his suggestion to have $\log 1 = 0$ and that the logarithm of 10 should be a power of 10 (not necessarily unity) but even with this beginning they did not use the notation of exponents.

APPENDIX B

Note on Rationals and Transcendentals

It will be recalled that a number is rational if it can be represented as the ratio of two integers, otherwise the number is irrational. Furthemore if an irrational can be represented as the root of a polynomial then it is an algebraic irrational, otherwise it is a transcendental. Hence $\sqrt{2}$ is irrational as it can't be represented as the ratio of two integers. But it is not transcendental as it is the root of the polynomial $x^2 - 2 = 0$. On the other hand it can be shown that the irrational e cannot be represented as the root of such an equation no matter how large the degree and hence it must be transcendental.

It was first shown in 1761 by Lambert (1728-77) that both e and π were irrational. It took over a century to show that these numbers were transcendental. Hermite (1822-1905) proved that e was transcendental in 1873 (whilst Lindemann (1852-1939) proved in 1882 that π was transcendental).

Hermite (1822-1901) also showed that e^y was irrational for rational y but it was not until 1929 that Gelfond showed that e^π was irrational. It is interesting to note that π^e has still not been shown to be irrational. Even more surprising is that the irrationality of γ, Euler's constant (see the main theorem), is still undecided! For the proof that e is irrational see Q9.

APPENDIX C

The Age Dependent Population Model

We now consider a fairly realistic population model in which the birth and death parameters are age dependent. In what follows all individuals are females, t refers to time and x to age. So we define

$b(x)\Delta x =$ the probability that an individual of age x gives birth in the interval x to $x + \Delta x$.

$P(x) =$ the survivor function - which is the probability that a new born individual will survive beyond x.

$B(t) =$ total birth rate.

Now, individuals of age x to $x + \Delta x$ must have been born at $t - x$, $t - x - \Delta x$ and survived a period $t - x$. That is $B(t-x)P(x)\Delta x$ must have survived to age x and thus the contribution to the total birth rate is $B(t-x)P(x)b(x)dx$.

Summing over all ages

$$B(t) = \int_0^\infty B(t-x)b(t)P(t)dt$$

Now this is called an integral equation in the unknown $B(t)$ as this term occurs within the integral as well as on the left hand side.

It is known that for small t the solution is often the sum of many exponentials. What is more remarkable is that the solution when t is large always approaches the single exponential

Ae^{rt},

where r is the largest root of

$$\int_0^\infty e^{-rt} b(t)P(t)dt = 1.$$

This is the justification for claiming that e is indeed the magic number of growth!

APPENDIX D

The following is an extract from the book Feynman, Richard P., “SURELY YOU’RE JOKING, MR. FEYNMAN!" Adventures of a Curious Character, as told to Ralph Leighton. Ed. Published by Edward Hutchings, W.W Norton and Company, New York, London. (also in softback by Bantam Books, New York).

Richard Feynman who died in 1989, apart from being one of the most original thinkers in and out of physics since the last world war, was undoubtedly one of the great eccentrics of the American University scene. Unlike many European University eccentrics many of whom had a cruel streak he was very lovable and made fun of his pompous targets in a gentle way.

Working on the atom bomb at Los Alamos he communicated with his wife in code and opened all the top security safes - he wrote up some of his Nobel prize winning physics in topless bars and had all sorts of remarkable encounters with gangsters - his unique ability to smash through cant and can’t, and go directly to the core of a problem allowed him as a member of the commission for the Challenger disaster to demonstrate very dramatically that the cause was simply due to an O-ring failing because of low temperatures.

On his death his students at Caltech raised a large banner inscribed with their final testimony to him - “We love you Richard".

So if you can’t do the sums that follow as quickly as Feynman, console yourself with the awareness that you are being given a lesson from a genius.

By the way the Marchant he refers to is a very old fashioned semi mechanical calculator where you had to rotate a handle to do multiplications.

The following tells you something about e but even more importantly it tells you a great deal about the difference between being able to merely manipulate numbers, and understanding the nature of numbers. Whatever you do buy the book, read it for enjoyment and at the same time you will learn about one of the most important lessons that there is to know in science: the pleasure of playing with a project. The value of this neglected activity is shown as follows - Feynman is in the cafeteria and observes that a plate thrown by a student wobbled in an interesting way. He then returns to his office and “played around" with some equations relating to wobbles with no thought about a research plan “The diagrams and the whole business that I got the Noble prize for came from that piddling around with the wobbling plate."

LUCKY NUMBERS

One day at Princeton I was sitting in the lounge and overheard some mathematicians talking about the series for e^x, which is $1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots$. Each term you get by multiplying the preceding term by x and dividing by the next number. For example, to get the next term after $x^4/4!$ you multiply that term by x and divide by 5. It’s very simple.

When I was a kid I was excited by series, and had played with this thing. I had computed e using that series, and had seen how quickly the new terms became very small. I mumbled something about how it was easy to calculate e to any power using that series (you just substitute the power for x).

“Oh yeah ?" they said, “Well, then, what’s e to the 3.3?" said some joker - I think it was Tukey.

I say, “That’s easy. It’s 27.11."

Tukey knows it isn’t so easy to compute all that in your head. “Hey! How’d you do that?"

Another guy says, “You know Feynman, he’s just faking it. It’s not really right."

They go to get a table, and while they’re doing that, I put on a few more figures: “27.1126," I say.

They find it in the table. “It’s right! But how’d you do it!"

“I just summed the series."

“Nobody can sum the series that fast. You must just happen to know that one. How about e to the 3?"

“Look," I say. “It’s hard work! Only one a day!"

“Hah! It’s a fake!" they say, happily.

“All right," I say, “It’s 20.085."

They look in the book as I put a few more figures on. They’re all excited now, because I got another one right.

Here are these great mathematicians of the day, puzzled at how I can compute e to any power! One of them says, “He just can’t be substituting and summing - it’s too hard. There’s some trick. You couldn’t do just any old number like e to the 1.4."

I say, “It’s hard work, but for you, OK. It’s 4.05."

As they’re looking it up, I put on a few more digits and say, “And that’s the last one for the day!" and walk out.

What happened was this: I happened to know three numbers - the logarithm of 10 to the base e (needed to convert numbers from base 10 to base e), which is 2.3026 (so I knew that e to the 2.3 is very close to 10), and because of radioactivity (mean-life and half-life), I knew the log of 2 to the base e, which is .69315 (so I also knew that e to the .7 is nearly equal to 2). I also know e (to the 1), which is 2.71828.

The first number they gave me was e to 3.3, which is e to the 2.3 - ten - times e, or 27.18. While they were sweating about how I was doing it, I was correcting for the extra .0026 - 2.3026 is a little high.

I knew I couldn’t do another one; that was sheer luck. But then the guy said e to the 3:

that's e to the 2.3 times e to the .7, or ten times two. So I knew it was 20 something, and while they were worrying how I did it, I adjusted for the .693.

Now I was sure I couldn't do another one, because the last one was again by sheer luck. But the guy said e to the 1.4, which is e to the .7 times itself. So all I had to do is fix up 4 a little bit!

They never did figure out how I did it.

APPENDIX E

From the book "Memorabilia Mathematica" by Robert Edouard Moritz (Mathematical Association of America). (see review by Keith Tognetti, Gazette, Austr. Math. Soc., Vol 23 No 4 , Nov 96, p177).

This is an unaltered and unabridged republication of the first edition of 1914. The main purpose is to seek out exact statements and references to famous passages about mathematics and mathematicians - this is the work of a genuine philomath which in more enlightened times meant simply a lover of learning (from the Greek mathema which originally meant learning - only later was it taken over to apply only to maths).

Surely even an economic rat can see that there is something wrong with his sums when he reads this.

(Item 2130) "If the Indians hadn't spent the $24". In 1626 Peter Minuit, the first governor of New Netherland, purchased Manhattan Island from the Indians for about $24 $\ldots$ assume for simplicity a uniform rate of 7% (for the 280 years, this percentage is about average for the stockmarket when it is not crashing) then the value in 1906 would be $24 * 1.07^{280}$ = more than $4,042 million. This turned out to be enough to buy the entire borough back again!

Furthermore if the Indians came to collect on their money today its value would be over 2 trillion dollars!!!

So much for sustainable interest rates and economics being an exact science.