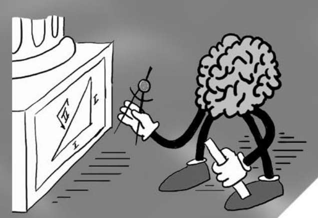

A dense distance 1 excluding set in $\mathbb{R}^3$

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Abstract

The study of the chromatic number of Euclidean space leads to the question of the maximum overall density that a single distance 1 excluding set can attain. In 1967 Croft presented a construction of what is still the densest known distance excluding set in $\mathbb{R}^2$. In this paper an analogous construction for $\mathbb{R}^3$ is presented which is the densest known distance excluding set in $\mathbb{R}^3$. A general approach for $\mathbb{R}^n$ is also discussed. The implications of the possibility that this is the densest set (or close to it) are discussed, particularly with regards to the lower bound for the chromatic number of $\mathbb{R}^3$ using measurable colourings.

Introduction

The chromatic number $\chi(\mathbb{R}^n)$ of Euclidean n-space is the minimum number of colours required to colour each point of the space such that no two points that are distance 1 apart receive the same colour. (We say the colouring *excludes distance 1*.)

The only known value is $\chi(\mathbb{R}^1) = 2$ as can be seen by $\mathbb{R} = \bigcup_{n\in\mathbb{Z}}[n, n+1)$ with the colouring

$$\text{colour}(x) = \begin{cases} 0 & \lfloor x \rfloor \text{ is odd,} \\ 1 & \lfloor x \rfloor \text{ is even,} \end{cases}$$

where $\lfloor x \rfloor$ is the greatest integer $\leq x$.

For other values of n only upper and lower bounds are known. The type of colouring can be restricted (e.g. to measurable sets (χ_m) or polygon tilings, etc.) and in many cases the lower bound can be improved under these restrictions [9], [13].

A related question is that of the greatest density ρ that a single colour can have in the space and be distance 1 excluding [12]. This question is of course restricted to colouring by measurable sets. Clearly $\chi_m \geq \rho^{-1}$. The set constructed in this paper gives a new lower bound for $\rho(\mathbb{R}^3)$.

A summary of some of the known bounds on χ, χ_m and ρ in low dimensions is given in Table 1.

The densest known set in $\mathbb{R}^2$ was constructed by Croft [3] and is depicted in Figure 1. It consists of a figure that is the intersection of a hexagon and a circle of slightly smaller diameter placed on the points of the equilateral triangle lattice. In other words, the figure

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Table 1.

$\mathbb{R}^2$	$\chi \geq 4$ [1]	$\chi_m \geq 5$ [4]	$\rho \leq \frac{12}{43}$ [11]	$\chi \leq 7^{(1)}$
$\mathbb{R}^3$	$\chi \geq 6$ [6]	$\chi_m \geq 6$ [4]	$\rho \leq \frac{7}{37}$ [10]	$\chi \leq 15$ [2]
$\mathbb{R}^4$	$\chi \geq 6$ [1]	$\chi_m \geq 7$ [4]	$\rho \leq \frac{4}{23}$ [5]	$\chi \leq 49^{(2)}$

$^{(1)}$Demonstrated by a well known colouring based on regular hexagons.
$^{(2)}$Demonstrated by a lattice/sublattice colouring in $\mathbb{R}^4$.

is a circle with six chords cutting off six equal segments. It is not difficult to confirm that in the optimal configuration the chords subtend an angle of 0.5266 (a transcendental multiple of π) radians at the centre and the density achieved is 0.2294 (thus $\rho(\mathbb{R}^2) \geq 0.2294$).

Heuristically we may imagine the process that leads to this construction as follows:

(1) Centre open circles of small radius on points distance 2 apart, arranged according to the equilateral triangle lattice.
(2) Increase their radius uniformly. We may do this until the circles have radius $\frac{1}{2}$ at which time 1 is the unique excluded distance.
(3) Continue to increase the area of the 'circles' by expanding towards the centres of the triangles but drawing the circle in from neighbouring circles in order to maintain an excluded distance.

The process is illustrated in Figure 1.

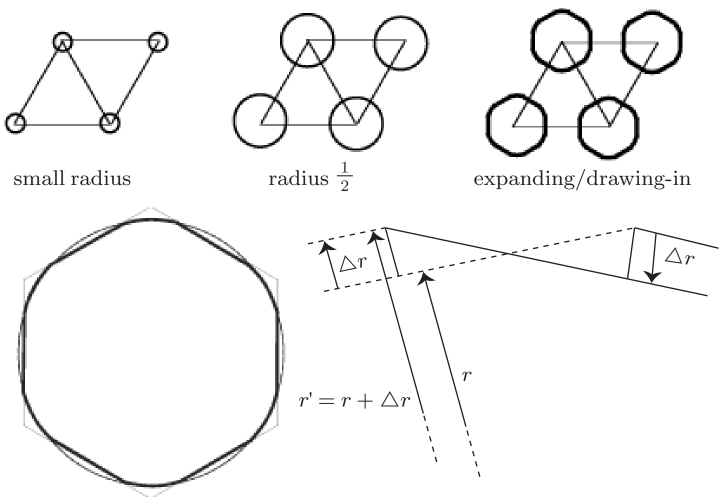

Figure 1. Top: the 'circle expansion' process with resultant hexagonal dice. Bottom: local change in a die with a change in the radius of the curved arcs, r (the 'curved' section of perimeter is dashed). The change in area will be (straight perimeter$-$curved perimeter)$\times\triangle r$+$O((\triangle r)^2)$.

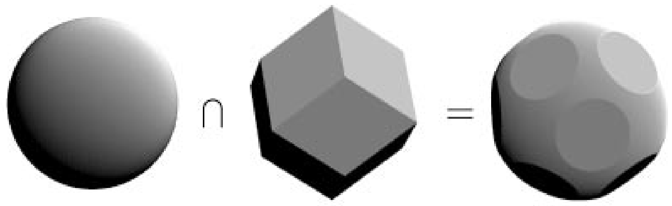

Figure 2. The Rhombic Dodecahedral Die of diameter $2r_s$, ($\mathrm{RDD}(r_s)$) is the intersection of a sphere of radius r_s, $S(r_s)$ and scaled rhombic dodecahedron $(1-r_s)\mathcal{V}$.

We call the figures formed by this process 'hexagonal dice'. A neat way to characterise the dimensions of the hexagonal die of maximal area is to note that it occurs when the lengths of the chords and curved arcs are equal. To see this suppose that the hexagonal die has area A. If we increase the radius of the curved arcs by $\triangle r$ then the chords must be drawn in by $\triangle r$ to maintain an excluded distance. So

$$\triangle A = (\triangle r) \times L(\text{curved perimeter}) - (\triangle r) \times L(\text{straight perimeter}) + O((\triangle r)^2),$$

where $L(\ )$ is length. Thus the area is maximised when the lengths are equal.

This technique for creating a dense set can be generalised to $\mathbb{R}^n$ and the purpose of this paper is to present an analogous construction in $\mathbb{R}^3$. The general method is to start with the best known lattice based sphere packing for $\mathbb{R}^n$. By halving the radius of the n-spheres we can immediately produce quite a dense distance excluding set (with open spheres). The next step is to improve on this density by 'expanding' towards the lattice holes and 'drawing in' from the direction of neighbouring spheres. More precisely, we take the intersection of an n-sphere and a scaled Voronoi region (nearest neighbour region) of the lattice. The Voronoi region is a polytope with faces perpendicular to the lattice vectors. Hence the resulting figure is an n-sphere with flat faces sheared off, somewhat resembling a many-sided n-dimensional die. We will see below (as a check on calculations) that the equal curved and flat surface area characterisation holds for $\mathbb{R}^3$, and indeed in general $V_n(\text{die}) = \triangle r \times (V_{n-1}(\text{'curved part'}(\partial(\text{die}))) - V_{n-1}(\text{'flat part'}(\partial(\text{die})))) + O((\triangle r)^2)$ as the 'curved part' of the boundary is pushed out and the 'flat part is drawn in. Though this method can be applied to any underlying lattice it is not possible to give a general construction explicitly as each case depends on the packing lattice used.

The construction in $\mathbb{R}^3$

The approach is to use the face centred cubic lattice FCCL[1] (the best sphere packing lattice for $\mathbb{R}^3$) scaled so that distinct lattice points are at least 2 units apart, that is

$$\mathrm{FCCL} = \{z_1(\sqrt{2},\sqrt{2},0) + z_2(0,\sqrt{2},\sqrt{2}) + z_3(\sqrt{2},0,\sqrt{2}) \colon z_i \in \mathbb{Z}\}.$$

We centre open spheres of radius $\frac{1}{2}$ on these lattice vectors and 'expand' towards holes $r_s = \frac{1}{2} + \triangle$ and 'draw in' $r_d = \frac{1}{2} - \triangle$ from lattice vectors distance 2 away. This forms a rhombic dodecahedral die (RDD) that has $2r_s$ as an excluded distance. Here r_s and r_d are the maximum and minimum radii of the RDD respectively.

[1] Also known as $D_3 = \{\boldsymbol{v} \in \mathbb{Z}^3 \colon v_1 + v_2 + v_3 \in 2\mathbb{Z}\}$ or $A_3 = \{\boldsymbol{w} \in \mathbb{Z}^4 \colon w_1 + w_2 + w_3 + w_4 = 0\}$.

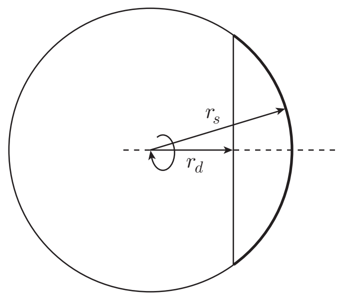

Figure 3. The 'caps' cut from the sphere $S(r_s)$.

More precisely the holes of FCCL are the vertices of the Voronoi region $\mathcal{V}$ of FCCL and their translates by vectors in FCCL. The Voronoi region $\mathcal{V}$ is a rhombic dodecahedron and is the convex hull of its vertices $\sqrt{2}\{\pm(1,0,0), \pm(0,1,0), \pm(0,0,1), (\pm\frac{1}{2}, \pm\frac{1}{2}, \pm\frac{1}{2})\}$ with volume $4\sqrt{2}$.

Maintaining $2r_s$ as an excluded distance where $2r_s$ is the diameter of the RDD means the RDD is the intersection of an open sphere of radius r_s and an open $r_d\mathcal{V}$ as pictured in Figure 2.

If $\frac{1}{2} \leq r_s \leq 4 - 2\sqrt{3}$ (so the sphere is inside the edges of the r_d scaled copy of $\mathcal{V}$) then the RDD will be a sphere with twelve identical circular 'caps' cut off by the faces of the rhombic dodecahedron $r_d\mathcal{V}$. This makes the calculation of its volume quite simple since the caps are just solids of rotation, see Figure 3.

$$\begin{aligned} V_{\mathrm{cap}(r_s)} &= \pi \int_{r_d}^{r_s} r_s^2 - x^2 \, dx \\ &= \frac{\pi}{3}[4r_s^3 - 3r_s + 1] \quad \text{as } r_s + r_d = 1. \end{aligned}$$

Thus (for $\frac{1}{2} \leq r_s \leq 4 - 2\sqrt{3}$)

$$\begin{aligned} V_{\mathrm{RDD}(r_s)} &= V_{S(r_s)} - 12 \times V_{\mathrm{cap}(r_s)} \\ &= \frac{4\pi}{3}(9r_s - 11r_s^3 - 3) \end{aligned}$$

On the closed interval $\frac{1}{2} \leq r_s \leq 4 - 2\sqrt{3}$ the maximum volume of the RDD is

$$V^*_{\mathrm{RDD}} = V_{\mathrm{RDD}(\sqrt{3/11})} = \frac{4\pi}{3}\left(6\sqrt{\frac{3}{11}} - 3\right) \approx 0.5588,$$

given when $r_s = \sqrt{3/11}$.

Note also that for a rhombic dodecahedral die of (locally) maximal volume the 'flat surface area' will be equal to the 'curved surface area' (consider as in $\mathbb{R}^2$ referring to Figure 1, an infinitesimal change in r_s when volume is a local maximum). Clearly the proportion of 'curved surface area' to 'flat surface area' is a decreasing function of r_s thus there is a unique RDD giving a local maximum volume (when $r_s = \sqrt{3/11}$) and this must be the global maximum as $V_{\mathrm{RDD}(0)} = V_{S(0)} = 0$, $V_{\mathrm{RDD}(1)} = V_{(1-1)\mathcal{V}} = 0$ and $V_{\mathrm{RDD}(r_s)}$ is a continuous function of r_s.

We now verify that when $r_s = \sqrt{3/11}$ for the RDD we have equality between 'curved surface area' and 'flat surface area'.

The surface area of the curved part of the cap is

$$\int_0^{2\pi} \int_0^{\arccos((1-r_s)/r_s)} r_s^2 \sin\phi \; d\phi \; d\theta = 2\pi(2r_s - 1)r_s.$$

Thus for $r_s = \sqrt{\frac{3}{11}}$,

$$\begin{aligned} \text{'flat area'} &= 12 \times \pi(r_s^2 - r_d^2) \\ &= 4\pi r_s^2 - 12 \times (2\pi(2r_s - 1)r_s) \\ &= \text{'curved area'}. \end{aligned}$$

The maximum volume for the rhombic dodecahedral die V^*_{RDD} gives a density of

$$\frac{V^*_{\mathrm{RDD}}}{(V_{\mathcal{V}})} = \frac{\pi}{3\sqrt{2}}\left(6\sqrt{\frac{3}{11}} - 3\right) \approx 0.09878.$$

The density of the set of open spheres of radius $\frac{1}{2}$ placed on the same lattice will be $4\pi/3 \times 32\sqrt{2} \approx 0.09256$. Hence the density of the RDD based set is about 6.7% greater than that of the sphere based set.

The importance of this result

The best known general lower bound for the chromatic number of n-space using measurable sets is $\chi_m(\mathbb{R}^n) \geq n + 3$ as demonstrated by Falconer [4].

The hexagonal die constructed by Croft [3] in $\mathbb{R}^2$ has a density between $\frac{1}{4}$ and $\frac{1}{5}$. If this were the densest possible set, and therefore no set was as dense as $\frac{1}{4}$, it would give a second proof of the fact that $\chi_m(\mathbb{R}^2)$ is greater than or equal to 5.

The case in $\mathbb{R}^3$ is considerably more interesting. Falconer's result gives a lower bound of 6 for $\chi_m(\mathbb{R}^3)$, but the density of the construction presented here is between $\frac{1}{10}$ and $\frac{1}{11}$ (as is that of the plain sphere based set). If this were the densest possible set in $\mathbb{R}^3$ we would have a proof that $\chi_m(\mathbb{R}^3) > 10$ — quite an improvement.

Open questions

We have only considered sets based on a regular lattice, and only figures consisting of flat faces and spherical sections. Could there not be an irregular set which achieved a greater density? Certainly there could, but these sets are tremendously more difficult to analyse. Clues as to whether irregular sets might offer higher densities could be sought first in the two dimensional setting.

Acknowledgement

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