Plane quadrilaterals

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Abstract

The various types of plane quadrilaterals are characterized by their side and diagonal lengths. Pantographs are described. The set of all congruence classes of quadrilaterals is a variety of degree six in E^6.

There are in Euclidean plane geometry some elegant theorems about cyclic quadrilaterals, most of them consequences of the angle properties in circles. For example, there is Ptolemy's theorem[1]: *In a cyclic quadrilateral, the product of the diagonals equals the sum of the products of opposite sides.* For noncyclic quadrilaterals, results are not so easily come by. One general theorem of note is Euler's[2]: *In any quadrilateral, the sum of the squares on the four sides is equal to the sum of the squares on the diagonals plus four times the square on the line joining the midpoints of the diagonals.* Euclid, in *The Elements*, devotes no more than passing interest in quadrilaterals which are not regular in some way, such as parallelograms. But later geometers have of course filled this gap, so that by the end of the nineteenth century the trigonometric properties of general quadrilaterals appear as suitable material for school textbooks, see [2], [1]. Nevertheless the emphasis was on cyclic quadrilaterals.

Plane quadrilaterals can be classified into *types* as follows[3] (see also Figure 1):

- *convex,* in which the two diagonals are internal and intersect;
- *nonconvex and not selfintersecting* ('dart'), in which one diagonal is internal and one external, the two not intersecting, the external diagonal spanning the concavity;
- *selfintersecting* ('zigzag'), in which one pair of opposite sides intersect, and both diagonals are external to the contained area and do not intersect;
- *(partially) degenerate*, in which a particular two adjacent sides lie in the same line (here there are three types as shown in Figure 1: a 'flag', where one vertex is an internal point of a side and two sides overlap, a 'triangle', where two adjacent sides are in one straight line but not overlapping, and a 'bent line', where two opposite vertices coincide);
- *fully degenerate*, in which the whole figure is contained in one dimension.

This classification results from the definition of a quadrilateral as a plane figure determined by four points and four line segments, each point being an endpoint of exactly two segments and each segment having each of its endpoints at a point. We shall not distinguish between

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[1] Ptolemy, Claudius, 2nd century AD. The converse also holds. See [3, pp. 225–227], where Lachlan's proof of the converse is also given.

[2] See [3, Vol. 2, pp. 401–402].

[3] Euclid, Book 1, contains a different classification in terms of degrees of symmetry, as squares, rhombuses, . . . : see [3, Vol. 1, pp. 188, 189].

congruent quadrilaterals. The term 'convex' will conventionally be kept for the nondegenerate types. There are also several other degenerate types in which not all sidelengths are nonzero, but we will exclude these from consideration. So it is assumed hereafter that all sidelengths (but not necessarily diagonal lengths) are positive.

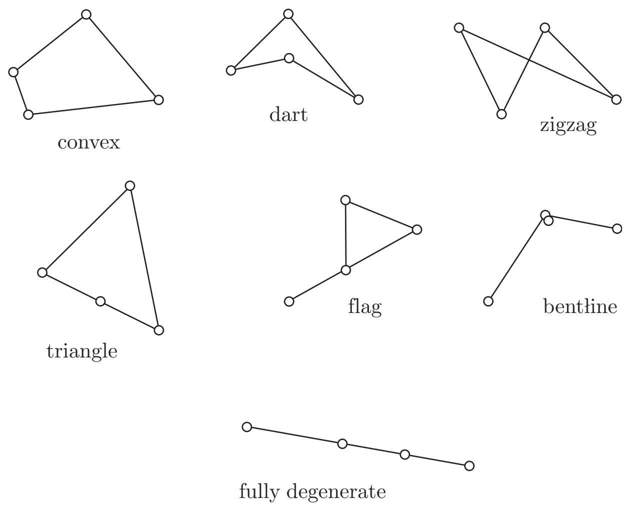

Figure 1. Types of plane quadrilaterals

It is easily proved that the four sidelengths and the two diagonal lengths together determine a quadrilateral uniquely. The four sidelengths and one diagonal length determine the quadrilateral to be one of two possibilities.

When can four given numbers be the side lengths of a quadrilateral? We find that: given an ordered tuple (a, b, c, d) of four positive numbers, a necessary and sufficient condition for the existence of a *convex* quadrilateral having those sidelengths, taken in that order, is:

$$\text{each number is less than the sum of the remaining three.} \tag{1}$$

For these inequalities clearly hold for the side lengths of any convex quadrilateral. Conversely, suppose a, b, c, d satisfy (1). Then

$$\max(a - b, b - a, c - d, d - c, 0) < \min(c + d, a + b),$$

so there exists a number x in this interval. The numbers a, b, x are the sidelengths of a nondegenerate triangle, so also are the numbers x, c, d, and these two triangles drawn back-to-back with common side x give a convex quadrilateral, one of whose diagonals has length x. Another solution, which is either a dart or a zigzag, results from drawing the triangles on the same side of side x.

The condition (1) is independent of the order of the tuple, so if there exists a convex quadrilateral with those side lengths in one order, there exist quadrilaterals for all other orders.

It is also true but a deeper result, using the intermediate value theorem, that (1) is necessary and sufficient for the existence of a convex cyclic quadrilateral with those sidelengths.

To see that (1) is also the appropriate condition for quadrilaterals of the other nondegenerate types, we introduce the notion of a *flexure.* This is a continuous operation which preserves the sidelengths of the quadrilateral and their order, but changes its shape. (It cannot in general be presented as a linear transformation of the plane in which the quadrilateral is embedded. We make the notion of continuity more precise presently.) With given sidelengths, the shape is a continuous function of an angle, say the angle between two chosen adjacent sides, or the angle between the diagonals, or of a diagonal length. By showing when each of the other forms can be reached from a convex form by flexing we can conclude that (1) is sufficient for that form (it is clearly necessary). If quadrilateral Ω can be flexed to Ω' we write $\Omega \sim \Omega'$.

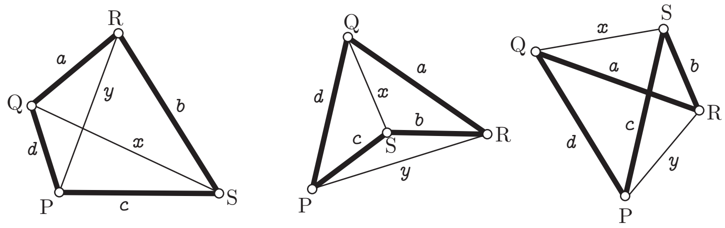

Figure 2. Notation for vertices and side and diagonal lengths

Therefore, let Ω be a convex quadrilateral with vertices P, Q, R, S and side lengths a, b, c, d, with $a = \mathrm{QR}$, etc. (see Figure 2, left). Suppose it is not a parallelogram: then there exists a pair of adjacent sides whose sum of lengths is less than the sum of lengths of the other pair of sides, say $a + b < c + d$. It can be flexed to a degenerate form which is a triangle Ω' say, in which the sides with lengths a and b are in the same line, meeting in the vertex R. Then a small further flexure to move R to the interior of Ω' is possible, giving a dart Ω''. Thus every quadrilateral which is not a parallelogram can be flexed to a dart.

To discuss flexure to a zigzag is less straightforward and depends upon a consideration of cases. Suppose $a > b > c > d$. We can fold side b onto side a to produce a flag Ω''' provided $c < (a - b) + d$. If

$$c \geqslant (a - b) + d, \tag{2}$$

then we try instead folding c onto b, which is possible provided $a < (b - c) + d$. If

$$a \geqslant (b - c) + d, \tag{3}$$

we try d onto c. If that does not work we try d onto a. Now the condition that all these four folds are impossible is a set of four inequalities; but these together lead to the contradiction $d \geqslant c$. Thus at least one of the folds is possible, and Ω is flexible to a flag. The case $a > b > c > d$ is one of 24 cases, which can be reduced to six needing separate consideration. After discussion of these, the end conclusion is given in Lemma 1.

Lemma 1. *Let Ω be a convex quadrilateral with sidelengths a, b, c, d in that order. Then Ω can be flexed to a flag in all cases except the following, in each of which Ω is flexible to a fully degenerate.*

(i) *Ω is a parallelogram.*
(ii) *The lengths of one pair of opposite sides are equal, and equal to the average of the lengths of the other two sides.*
(iii) *The lengths of one pair of adjacent sides are equal, and the lengths of the other pair are equal; Ω is a kite.*

Now if Ω can be flexed to a flag, then it can be flexed to a zigzag: for a small flexure of any flag to a zigzag is clearly possible, and flexure is an equivalence relation. Thus Theorem 1 follows.

Theorem 1. *The condition* (1) *holds for the sidelengths of any non fully-degenerate quadrilateral. It is a sufficient condition for the existence of a quadrilateral having those sidelengths, in that order, in the following classes:*

- $\mathbf{C}$*, the convex quadrilaterals,*
- $\mathbf{D} \cup \mathbf{P}$*, darts and parallelograms,*
- $\mathbf{Z} \cup (\sim \mathbf{Y})$*, zigzags and quadrilaterals which are flexible to fully degenerates.*

Here we have written $\mathbf{C}$ for the set of all convex quadrilaterals, $\mathbf{D}$ the darts, $\mathbf{P}$ the parallelograms, $\mathbf{Z}$ the zigzags, and $\mathbf{X}$ the set consisting of all degenerates, and $\mathbf{Y}$ the fully degenerates, with $\mathbf{Y} \subset \mathbf{X}$. Later we write $\mathbf{F}$ for the set of all flags, $\mathbf{T}$ the triangular degenerates. Note that $\mathbf{P} \subset (\sim \mathbf{Y})$.

Notation and topology

Given a quadrilateral Ω with sidelengths a, b, c, d in that order and diagonal lengths x, y, where x spans sides a, b and also c, d, we call

$$p = (a, b, c, d; x, y)$$

a *presentation* of Ω, and write $\Omega = \Phi(p)$ or sometimes $\Omega \equiv p$. We regard two quadrilaterals Ω, Ω' as equivalent and write $\Omega \equiv \Omega'$ if they are congruent figures, allowing congruence to include reversed orientation in the plane, and write $\mathbf{Q}$ for the set of all equivalence classes. Note that Ω has eight presentations $\sigma^j \tau^k p$ with $j = 0, 1, 2, 3$ and $k = 0, 1$, where σ and τ are the permutations

$$\sigma\colon (a, b, c, d; x, y) \mapsto (b, c, d, a; y, x) \text{ and } \tau\colon (a, b, c, d; x, y) \mapsto (d, c, b, a; x, y), \tag{4}$$

for which $\sigma^4 = \tau^2 = 1$, $\sigma\tau = \tau\sigma^3$, so that σ, τ are generators of a dihedral group. Thus $\Omega \equiv \Omega'$ means $\Omega \equiv p$, $\Omega' \equiv p'$ and $p = \sigma^j \tau^k p'$ for some j and k. Note the convention of listing in p the sides in order. It is necessary to introduce the notion of presentation because no satisfactory convention exists for all quadrilaterals to say with which side a listing of sidelengths should begin. (It is true that conventions can be invented for flags, for triangles and for darts, but that is not enough.) Write P for the set of all presentations.

Since p is a vector in Euclidean space $\mathrm{E}^4 \times \mathrm{E}^2$ we can use the Euclidean norm

$$\|\Omega\| = \|p\| = \sqrt{a^2 + b^2 + c^2 + d^2 + x^2 + y^2} \tag{5}$$

also as a function on quadrilaterals, since it is independent of the presentation. However, it is not then a norm, since the Euclidean distance does depend upon the presentations. To overcome this we introduce the function

$$D(\Omega, \Omega') = \min(\|p - \sigma^i \tau^j p'\|\colon i = 0, 1, 2, 3;\ j = 0, 1), \tag{6}$$

noting that $\|p - p'\| = \|\lambda p - \lambda p'\|$ for any permutation λ, so D is independent of the presentations used, and well defined.

Lemma 2. *D is a metric on $\mathbf{Q}$.*

Proof. It is clear that D is symmetric, and $D(\Omega,\Omega')=0$ when $\Omega\equiv\Omega'$. Suppose also $\Omega''\equiv p''$. For any h,i,j,k we have

$$D(\Omega,\Omega'')\le\|p-\sigma^h\tau^i p''\|\le\|p-\sigma^j\tau^k p'\|+\|\sigma^j\tau^k p'-\sigma^h\tau^i p''\|.$$

Fix j,k so that the first term on the right is $D(\Omega,\Omega')$, then choose h,i so that the second term is $D(\Omega',\Omega'')$. This proves that D satisfies the triangle inequality. □

In the same way we can introduce the metric $E(\Omega,\Omega')=\min(\|p-\sigma^i p'\|\colon i=0,1,2,3)$ on $\mathbf{Q}^\dagger$, the set of congruence classes where congruence is defined so as not to include reversed orientation. In $\mathbf{Q}^\dagger$ the further convention can be used for convex quadrilaterals, darts and partial degenerates that the order of listing sidelengths is clockwise, passing the inside of the quadrilateral on the right; but no such rule is possible for zigzags.

A flexure in $\mathbf{Q}$ can now be defined more precisely as a path $\Theta=\Phi\circ\phi$ determined by a continuous presentation-valued function $\phi\colon[0,1]\to P$ such that $\phi(t)=(a,b,c,d;x(t),y(t))$ where a, b, c and d are constants satisfying (1) and $x(t),y(t)$ satisfy (15), (16) below. Then $D(\Theta(t),\Theta(t'))$ is the minimum of five values $\|\phi(t)-\lambda\phi(t')\|$, where λ can be any of: the identity, $\sigma,\sigma^3,\tau\sigma,\tau\sigma^3$. Under the additional assumption that for all t the numbers $a,b,c,d,x(t),y(t)$ are all distinct we can show that λ must be the identity if $|\,t-t'|$ is small enough, that is, there exists $\delta>0$ such that

$$D(\Theta(t),\Theta(t'))=\|\phi(t)-\phi(t')\|=\sqrt{[x(t)-x(t')]^2+[y(t)-y(t')]^2}\quad\text{if}\quad|t-t'|<\delta.\tag{7}$$

This remark uses the uniform continuity of ϕ; see [6, Theorem 4.19, p. 91]. It asserts a locally minimizing property of ϕ.

Formulae

We mention some useful formulae for $\Omega\equiv p=(a,b,c,d;x,y)$. Let θ be the (acute) angle between the diagonals, and A the area. For convex quadrilaterals and darts the area is defined to be the area of the connected inside of the quadrilateral; for zigzags it is the modulus of the difference of the areas of the two bounded enclosed regions. Then in all cases we find that

$$A=\tfrac12 xy\sin\theta,\tag{8}$$

$$2xy\cos\theta=|a^2-b^2+c^2-d^2|,\tag{9}$$

$$16A^2+(a^2-b^2+c^2-d^2)^2=4x^2y^2,\tag{10}$$

$$2x^2y^2=\pm\sqrt{H(a,b,x)\cdot H(c,d,x)}-x^4+Sx^2+(a^2-b^2)(c^2-d^2),\tag{11}$$

where $S=a^2+b^2+c^2+d^2$ in (11), H denotes the Heron polynomial (see below), and the sign $+$ is to be taken if Ω is convex. The other cases are discussed presently. Proofs of identities (8), (9), (10) and many others can be found in [2, pp. 24–32], and [5, pp. 200–205], [1, pp. 168–176]. As a statement of a Pythagorean triple, (10) is particularly elegant. However, there is nothing to say that A or xy is rational when the side lengths are.

The Heron function H is the fourth degree symmetric polynomial in three variables $H(a,b,c)=-a^4-b^4-c^4+2b^2c^2+2c^2a^2+2a^2b^2$, which appears in the Heron formula $\sqrt{s(s-a)(s-b)(s-c)}$ for the area of a triangle with sides a,b,c and semiperimeter $s=\frac12(a+b+c)$. The area of the triangle is $\frac{1}{4}\sqrt{H(a,b,c)}$ (the formula is now attributed to Archimedes; see [4, p. 103]). Now in the case of a convex quadrilateral the area is that of the union of two triangles with common base x, so we can write

$$A = \tfrac{1}{4}\sqrt{H(a,b,x)} + \tfrac{1}{4}\sqrt{H(c,d,x)}.$$

Then substitution in (10) and simplification leads to (11). This gives a formula for y in terms of the other five lengths.

The equation (11) depends upon type as well as presentation. Let Ω be given, so that with presentation p we have (11) which we rewrite as $(11)_p$; there is also the equation

$$2x^2y^2 = \pm\sqrt{H(b,c,y)\cdot H(d,a,y)} - y^4 + Sy^2 + (b^2-c^2)(d^2-a^2); \qquad (11)_{\sigma p}$$

but $(11)_{\sigma^2 p}$ and $(11)_{\tau p}$ coincide with $(11)_p$ so there are only two distinct equations. Now the sign to be given the root term in $(11)_p$, which is the sign of

$$m(p) = 2x^2y^2 + x^4 - Sx^2 - (a^2-b^2)(c^2-d^2), \qquad (12)$$

is determined by whether the area of the quadrilateral is calculated as a sum or difference of areas. From this we arrive at the following rule for deciding the type of a quadrilateral, given in terms of its presentation p as opposed to an identifiable figure in the plane.

Theorem 2. *A quadrilateral Ω, given by one of its presentations p, is*

(i) *convex if $m(p)$, $m(\sigma p)$ are both positive,*
(ii) *a dart if $m(p)$, $m(\sigma p)$ are of opposite signs,*
(iii) *a zigzag if $m(p)$, $m(\sigma p)$ are both negative,*
(iv) *a flag if one of $m(p)$, $m(\sigma p)$ is zero and the other is negative,*
(v) *a triangle if one of $m(p)$, $m(\sigma p)$ is zero and the other is positive,*
(vi) *fully degenerate if $m(p)$ and $m(\sigma p)$ are both zero.*

A zigzag and a dart each has a distinguished pair of sides, say the intersecting pair for a zigzag, or the sides of the concavity for a dart. The following theorem characterizes these distinctions, using in one case the permutation function

$$\gamma\colon (a,b,c,d;x,y) \mapsto (a,x,c,y;b,d), \mathrm{P} \to \mathrm{P}. \qquad (13)$$

Theorem 3. *Let Ω be the quadrilateral $\Phi(a,b,c,d;x,y)$.*

(i) *If Ω is a dart and $m(p) < 0$ (so that $m(\sigma p) > 0$), then the edges of its concavity have lengths a,b (if $a+b < c+d$) or c,d (if $c+d < a+b$).*
(ii) *If Ω is a zigzag (so that $m(p) < 0$ and $m(\sigma p) < 0$), then $m(\gamma p)$ and $m(\sigma\gamma p)$ have the same sign; if it is negative then sides with lengths a,c intersect, if positive then sides with lengths b,d intersect.*
(iii) *If Ω is a triangle or flag, one diagonal is the sum or difference respectively of two adjacent sides, and this characterizes those features.*

Proof. (ii) and (iii) follow easily from the geometry. (i) Since $m(p) < 0$, the two areas $\frac{1}{4}\sqrt{H(a,b,x)}$, $\frac{1}{4}\sqrt{H(c,d,x)}$ are to be subtracted in $(11)_p$. This implies that x is the length of the external diagonal, so a,b or c,d are the lengths of the concavity edges. The concavity and diagonal are a triangle inside the other triangle, so the result follows from a theorem about triangles. □

Pantograph

If the midpoints of adjacent sides of Ω are joined in order, we get a parallelogram — the *median parallelogram* of the quadrilateral. Its existence is a welcome oasis of symmetry in an otherwise disorderly figure. The sides of the parallelogram are parallel to the diagonals of Ω, so its angle between adjacent sides equals the angle between the diagonals of Ω, the side lengths are half the diagonal lengths of Ω, and its area is half A (see Figure 3).

Now suppose instead that we are given an arbitrary parallelogram $\Lambda = \mathrm{FGHI}$, with sides of lengths f, g inclined at angle ω. Let QR be any line segment whose midpoint is F, then construct QP, RS having midpoints I, G respectively. We find that PS has H as its midpoint: so PQRS is a quadrilateral whose median parallelogram is Λ. Any parallelogram Λ generates in this way a doubly infinite family $\mathbf{Q}(\Lambda)$ of quadrilaterals for each of which it is the median. We call this family (with some abuse of terminology) the *pantograph generated by* Λ. The family contains quadrilaterals of all nondegenerate types as well as triangular and flag degenerates; all have the same diagonal lengths $x = 2f, y = 2g$, the diagonals being inclined at the same angle ω, have the same area, and therefore by (9) have the same value for $|a^2 - b^2 + c^2 - d^2|$. (It is perhaps more appropriate to regard a pantograph as a phenomenon in $\mathbf{Q}^\dagger$ rather than $\mathbf{Q}$.)

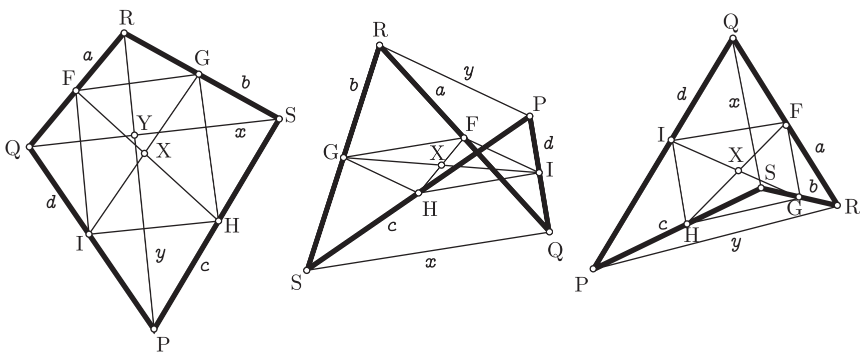

Figure 3. Quadrilaterals with their median parallelogram

If point P say traces out a figure Ψ, the other points Q, R, S trace out congruent figures, the figures of Q and S rotated through 180 degrees but with the same orientation. Here we see the action of a pantographic-like linkage. (More generally we can choose points F, G, H, I not as midpoints but as points of subdivision with specified ratio ρ, and obtain copies of Ψ with magnification ρ.)

If a and b are prescribed (subject to $a + b > 2f$) there are two quadrilaterals in $\mathbf{Q}(\Lambda)$ with those sidelengths for the sides through F and G, one of which is convex; so c and d are determined to that extent.

Theorem 4. *Let Λ be a parallelogram with sidelengths f, g and angle between the sides ω. Let Ω be a quadrilateral with diagonal lengths $2f, 2g$ and angle between the diagonals ω. Then $\Omega \in \mathbf{Q}(\Lambda)$.*

Proof. Construct Λ_1, the median parallelogram of Ω. Its sidelengths are f, g and the angles between its sides is ω. Thus $\Lambda_1 \equiv \Lambda$, and accordingly $\Omega \in \mathbf{Q}(\Lambda)$ since $\Omega \in \mathbf{Q}(\Lambda_1)$. (Recall that we identify congruent figures.) $\square$

Corollary 1. *Given the parallelogram* Λ *as above, a quadrilateral* $\Omega \equiv (a, b, c, d; x, y)$ *belongs to* $\mathbf{Q}(\Lambda)$ *if and only if*

$$|a^2 - b^2 + c^2 - d^2| = 8fg\cos\omega, \qquad x = 2f, \qquad y = 2g. \tag{14}$$

The space Q

We now give a geometric description of the space $\mathbf{Q}$, treating a, b, c, d, x, y as coordinates in $\mathrm{E}^6 = \mathrm{E}^4 \times \mathrm{E}^2$, and start with P, the space of presentations. P is contained in the first 2^6-ant of E^6; the inequalities such as $a < b+c+d$ describe open half-spaces, so condition (1) confines P to a subset of an open wedge with edge-face the xy-plane and bounded by four primes. The conditions

$$\max(|a-b|, |c-d|) \le x \le \min(a+b, c+d), \tag{15}$$

$$\max(|b-c|, |d-a|) \le y \le \min(b+c, d+a), \tag{16}$$

which clearly must hold, describe the intersection of 12 further half-spaces. Altogether P is in the region bounded by $6+4+12 = 22$ primes. From these we can drop $x = 0$, $y = 0$ since these boundaries are implied by (15), (16), and thus count 20 primes. This set is mapped onto itself by each of σ and τ. Condition (11), when cleared of the root sign and simplified, shows that P is the variety V whose equation is

$$x^2y^4 + B(x)y^2 + C(x) = 0, \tag{17}$$

where

$$B(x) = x^4 - (a^2 + b^2 + c^2 + d^2)x^2 - (a^2 - b^2)(c^2 - d^2), \tag{18}$$

$$C(x) = (b^2 - c^2)(a^2 - d^2)x^2 + (a^2c^2 - b^2d^2)(a^2 - b^2 + c^2 - d^2), \tag{19}$$

and is contained also in the variety V_σ got by applying σ to the equation for V. Note that the equation is invariant under τ. Thus P is the set obtained by intersecting the 20 half-spaces mentioned above with V, which is also their intersection with V_σ.

We now impose upon P the equivalence relation $\equiv$ where $p \equiv q$ if $p = \sigma^j\tau^k q$ for some $j = 0, 1, 2, 3; k = 0, 1$. $\mathbf{Q}$ is defined to be the set of equivalence classes $\mathrm{P}/\equiv$, with the metric topology induced by D. We leave it to the reader to verify Theorem 5.

Theorem 5. *The projection map* $\pi\colon \mathrm{P} \to \mathrm{P}/\equiv$ *(which coincides with the map* Φ *used earlier) is, with respect to the two metric topologies, both continuous and open, and therefore the metric topology of* D *on* $\mathbf{Q}$ *coincides with the quotient topology*[5]*.*

These are indeed serendipitous outcomes. $\mathbf{Q}$ consists of the four sets $\mathbf{C}$, $\mathbf{D}$, $\mathbf{Z}$, and $\mathbf{X}$. Write C for the set $\pi^{-1}(\mathbf{C})$, etc. It is clear that the shape of any quadrilateral is preserved under magnification, angles being preserved: this is the phenomenon of proportionality, and means that sets *C, D, Z, X* are cones pointed at the origin.

The sets $\mathbf{C}$, $\mathbf{D}$ and $\mathbf{Z}$ are open in $\mathbf{Q}$, the set $\mathbf{X}$ is closed, and contains $\mathbf{F}$ and $\mathbf{T}$. To prove the first three open we can argue from the geometry, or note that since the function m in (13) is continuous, the sets where m and $m \circ \sigma$ are of given signs are open, and invoke Theorem 2 and then the fact that π is an open map.

[5]See [7, Theorem 10.19, p. 62]. As a result about the projection map between metric spaces, this depends crucially upon (i) each equivalence class being finite, and (ii) the property here invoked as $\|\lambda p - \lambda p'\| = \|p - p'\|$ for all permutations λ.

To describe the connectivity of the space, write $\mathbf{Q}^* = \mathbf{Q} \setminus \mathbf{Y}$, where $\mathbf{Y}$ denotes as before the set of fully degenerates.

Theorem 6. *The sets* $\mathbf{C}$, $\mathbf{D}$, $\mathbf{Z}$, $\mathbf{F}$, $\mathbf{T}$ *are pathwise connected and hence connected subsets of* $\mathbf{Q}$.

Proof. We show that every Ω in $\mathbf{C}$ can be joined to the unit square by a path in $\mathbf{C}$. With Y being the point of intersection of the diagonals of Ω, let each vertex in turn be moved to the point on its diagonal at distance $1/\sqrt{2}$ from Y. Each such movement constitutes a path in $\mathbf{C}$, and their product path is therefore a path in $\mathbf{C}$ joining Ω to the unit square. This shows that $\mathbf{C}$ is pathwise connected. Similar proofs for the other sets can be constructed. □

We have, from Theorem 2 by similar arguments:

Theorem 7. *In the space* $\mathbf{Q}^*$ *with the relative topology,*

(i) $\mathrm{bdry}(\mathbf{C}) \cap \mathrm{bdry}(\mathbf{Z}) = \emptyset$,
(ii) $\mathbf{F} = \mathrm{bdry}(\mathbf{D}) \cap \mathrm{bdry}(\mathbf{Z}) \setminus \mathrm{bdry}(\mathbf{C})$,
(iii) $\mathbf{T} = \mathrm{bdry}(\mathbf{D}) \cap \mathrm{bdry}(\mathbf{C}) \setminus \mathrm{bdry}(\mathbf{Z})$,
(iv) *Every path joining* $\Omega \in \mathbf{Z}$ *to* $\Omega' \in \mathbf{C}$ *meets* $\mathbf{F}$*; every path joining* $\Omega \in \mathbf{Z}$ *to* $\Omega'' \in \mathbf{D}$ *meets* $\mathbf{F}$*; every path joining* $\Omega' \in \mathbf{C}$ *to* $\Omega'' \in \mathbf{D}$ *meets either* $\mathbf{F}$ *in at least two points, or* $\mathbf{T}$.

By Ptolemy's theorem and its converse, the cyclic quadrilaterals are represented by points on the intersection of P and the quadric $xy = ac + bd$. This intersection is a variety of dimension 4, degree 12.

For a given parallelogram $\Lambda \equiv (f, g, f, g;\ \mathrm{angle}\,\omega)$, the pantograph $\mathbf{Q}(\Lambda)$ is given by the intersection of P and the hypersurface (14), which is of dimension 3, degree 4. This intersection is thus of dimension 2, a surface of degree 12. The cyclic quadrilaterals in the pantograph therefore constitute a curve in E^6 of degree 24. It is a matter for further investigation, to discover how much of this curve is real, and whether the implied complexity is in fact a consequence of the representation.

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References

[1] Carslaw, H.S. (1945). *Plane Trigonometry.* London, Macmillan.
[2] Durell, C.V. and Robson, A. (1945). *Advanced Trigonometry.* London, Bell.
[3] Heath, T.L. (1956). *Euclid: The Thirteen Books of the Elements* (3 volumes), 2nd edn. New York, Dover Publications.
[4] Heath, T.L. (1981). *A History of Greek Mathematics*, Vol. 2. New York, Dover Publications.
[5] Hobson, E.W. (1891). *A Treatise on Plane Trigonometry.* C.U.P.
[6] Rudin, W. (1976). *Principles of Mathematical Analysis.* New York, McGraw-Hill.
[7] Thron, W.J. (1966). *Topological Structures.* New York, Holt, Reinhart and Winston.